Chance, Strategy, and Choice

Games and elections offer familiar, current, and lively subjects for a course in mathematics. With applications in a range of areas such as economics, political science, and sociology, these topics are of broad interest. This undergraduate textbook, primarily intended for a general-education course in game theory at the freshman or sophomore level, provides an elementary treatment of games and elections. Starting with basics such as gambling games, Nash equilibria, zero-sum games, social dilemmas, combinatorial games, and fairness and impossibility theorems for elections, the text then goes further into the theory with accessible proofs of advanced topics such as the Sprague–Grundy Theorem and Arrow's Impossibility Theorem. The book

- uses an integrative approach to probability theory, game theory, and social choice theory by highlighting the mix of ideas occurring in seminal results on games and elections such as the Minimax Theorem, allowing students to develop intuition in all areas while delving deeper into the theory;
- provides a gentle introduction to the logic of mathematical proof, thus equipping readers with the necessary tools for further mathematical studies, a feature not shared by most game theory texts;
- contains numerous exercises and examples of varying levels of difficulty to help students learn and retain the material;
- requires only a high school mathematical background, thus making the text accessible to a large range of students.

Samuel Bruce Smith is Professor and Chair of Mathematics at Saint Joseph's University and a past director of the University Honors Program. In 2012, he won the Tengelmann Award for Distinguished Teaching and Research.

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An Introduction to the Mathematics of Games and Elections

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To Patty, David, and Ned

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Part III Special Topics

Preface

This book was written to teach an accessible and authentic mathematics course to a general audience. Games and elections are ideal topics for such a course. The motivating examples are familiar and engaging: from the 2000 U.S. presidential election, to economic pricing wars, to gambling games such as Poker and Blackjack. The elementary theory of these topics involves manageable computations and emphasizes analytic reasoning skills. Most important, the study of games and elections has produced a wealth of beautiful theorems, many within the last century. Games and elections offer a wide selection of elegant results that can be approached without extensive technical background. A course on these topics provides a view of mathematics as a powerful and thriving modern discipline.

There are excellent texts covering aspects of the theory of games and elections, such as those of Straffin [48] and Taylor and Pacelli [49]. The topics are also frequently covered as chapters in liberal arts mathematics texts. This text is pitched at a level between these two types of treatments. The book is designed to be more mathematically ambitious than the general textbook but to allow for a less demanding course than one offered using a more specialized text.

A novelty of this book is the integrative approach taken to the three subjects: probability theory, game theory, and social choice theory. This approach highlights the mix of ideas occurring in seminal results on games and elections such as the Jury Theorem, the Minimax Theorem, and the Gibbard-Satterthwaite Theorem. On a practical level, the integrative approach allows for a more gradual development of material. Rather than taking a steep descent in one area, the chapters follow a spiraling path from chance to strategy to social choice and back again, exploring examples and developing techniques and intuitions in all areas while delving deeper into the theory of games and elections.

Structure of the Book

This book is divided into three parts. Part I introduces the main devices used in the text. These are the probability tree, the game tree, the payoff matrix, and the preference table. Expected values, dominated strategies, backward induction, and rational play are then defined for the various types of games. The idea of preference ballots is motivated with examples from sports and politics, and the basic voting methods are introduced.

Part I includes three chapters exploring two of the three notions of chance, strategy, and choice together. The paradox of the chair, the Gibbard-Satterthwaite Theorem, swing states in the Electoral College, the Jury Theorem, and a simplified version of Poker are introduced in these chapters. The final chapter of Part I presents Nash equilibria as a unifying concept, tying together several examples in the categories of zero-sum, partial-conflict, mixed-strategy, and electoral games.

Part II focuses more directly on the development of mathematical theory. A chapter on logic prepares the way, with topics from the first part of the book serving as representative examples of conditional and universal statements. The method of proof by induction for game trees is illustrated with a version of Zermelo's Theorem. Conditionals are explored further in the probability setting, while fairness criteria for voting methods provide exercises with universal statements and an exposure to the dichotomy between proofs and counterexamples in mathematics. Counting techniques are developed for computing the power index of a weighted voting system. The problem of determining weights for a yes-no system and the notion of a mathematical invariant are introduced, leading up to a statement of the Taylor-Zwicker Theorem.

Part II also includes a series of chapters exploring the various types of games. Counting techniques are applied again to compute probabilities with card hands in a chapter on gambling games. The game Craps shows the power of introducing a conditional, whereas a problem in Poker motivates Bayes' formula. A chapter on zero-sum games contains the proof of the 2×2 Minimax Theorem and introduces techniques for the $2 \times n$ case. A chapter on partial-conflict games explores backward induction through a variety of examples, including the Iterated Prisoner's Dilemma, the Chain Store Game, the Dollar Auction, and Bram's Theory of Moves. Next, a chapter on takeaway games introduces the Sprague–Grundy numbers and includes a proof of Bouton's beautiful solution of the game Nim. The final chapter in Part II is a return to the question of fairness for voting methods. Fairness criteria for social welfare methods including Arrow's famous Independence criteria are introduced. May's Theorem, a simplified version of Arrow's Theorem, and Sen's Impossibility Theorem are proved in this chapter.

Part III consists of five independent chapters offering possible enhancements to a basic course. A chapter on probability explores famous paradoxes ranging from the Monte Hall Problem to Bertrand's Paradox in geometry. Nim Misère, Chomp, and Hex are analyzed in a chapter on combinatorial games. The debate over fairness between Borda and Condorcet is explored, with the arguments for both sides interlaced with three theorems: a version of the Jury Theorem, a modern result of McGarvey's on the realization of social preference graphs, and Borda's Theorem from his original paper introducing the Borda count. The final two chapters focus on proofs of two beautiful results on games and elections. The Sprague–Grundy Theorem for combinatorial games is proved, introducing the notion of Nim sums. The game Hackenbush is analyzed as an application. The final chapter is devoted to a proof of Arrow's Impossibility Theorem. This chapter includes a brief discussion of the modern era of social choice theory with the proof of Sen's Coherence Theorem and the introduction of nonpreferential voting methods.

Designing a Course

The text was written to allow for flexibility in designing a course. I have taught a general-education course from this book for several years. I typically cover the first two parts of the text treading lightly on the discussion of induction in Chapter 10 and subsequent proofs by induction in Part II. On occasion, I have had time to add one of the chapters in Part III as a last topic. I have also taught an honors version of this course for a number of semesters. This course moved quickly through Part I and included one or two chapters in Part III according to student interest.

It is easy to design a course emphasizing one topic or another from the various areas. For students who have had a course in probability, Chapters 3 and 11 can be skipped. A course emphasizing social choice theory might omit the developments on combinatorial games in Chapter 17. Induction proofs beginning with Zermelo's Theorem and the Law of the Probability Tree in Chapters 10 and 11 can be omitted depending on the audience. The book can also be used for an elective course for sophomore-level mathematics majors. Such a course would emphasize the proof techniques including induction. The final chapters on the Sprague–Grundy Theorem and Arrow's Impossibility Theorem are worthy end goals for such a course.

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