1
Introduction

1.1 A tour guide of holographic matter

A quake is rumbling through the core of physics. Suddenly apparently unrelated areas appear to have a common ground, showing an eerie capacity to fertilise each other. In physics such occasions are invariably propelled by novel mathematical machinery and the present case is no exception. This new mathematical contraption is "holographic duality" (or "anti-de Sitter/conformal field theory correspondence"), which was originally discovered in string theory in the 1990s. Until recently its use was limited to the historic scope of string theory – particle physics and quantum gravity. At a breathtaking pace it has since rolled out over many of the subject areas of modern fundamental physics, even yielding new insights into old subjects such as the nineteenth-century theory of hydrodynamics.

Several books of this kind could be written, and are being written, highlighting how anti-de Sitter/conformal field theory (AdS/CFT) impacts on various fields in physics. This book will focus on a prominent area where the developments have been particularly stunning. This is the application to equilibrium condensed matter physics. This started in 2007, and in a matter of a few years condensed matter theory was rewritten in a different mathematical language. This language is the one that one would perhaps least expect: general relativity. On its own a rewriting of condensed matter might not sound like a great advance, no matter how unconventional the language. However, the correspondence makes it possible to explore regimes of quantum many-body physics that are completely inaccessible with conventional techniques. In particular we refer to non-Fermi-liquid states of matter formed in finite-density systems of strongly interacting fermion systems. The holographic mathematics here becomes particularly expressive, suggesting that a general principle of a new kind is at work. It appears that this principle relates to the physics of compressible quantum matter: the notion that the nature of this state of matter is governed by a macroscopic quantum entanglement involving all of its constituents. This discovery is not just remarkable on its own. "Holographic strange matter" also has tantalisingly suggestive resemblances to the mysterious phenomena observed experimentally in strongly interacting electron systems that have been realised in special materials such as the high-T_c superconductors. First seen around thirty years ago, these have defied any reasonable explanation despite countless attempts resting on the available mathematical techniques. Could it be that holography supplies the mathematical equations that will shine light on this number-one mystery of condensed matter physics?

The correspondence also has its limitations. Though much evidence has been collected in favour, the jury is still out both with regard to the quantum information aspects and regarding the empirical relevancy. At present holography is an exciting research programme with a potential to alter the fundaments of the theory of physics. This should be of interest to the physics community at large, but there is an intrinsic difficulty. String theory famously has to be a unified theory of physics, but by the discovery of AdS/CFT this was changed into the "unification of the *theories* of physics". The implication is that the physicist has to pay the price that he/she has to be expert in all fanciful, modern areas of physics at the same time! A head-on encounter with the application of holography often entails a nearly seamless switch between string theory and high-energy-style quantum field theory, sophisticated general relativity including the latest in black-hole physics, via modern condensed matter theory, all the way to the tedious data sorting that one encounters in dealing with the real world studied by experimental physics.

Our aim has been to lower the barrier of entry to this field and make it readily accessible. At present the required knowledge is still predominantly scattered throughout a hard-to-penetrate research literature. We have done our best to write a comprehensive overview of the main current of anti-de Sitter/condensed matter theory (AdS/CMT). It is not exhaustive: we had to make choices, and we hope that expert readers will agree with our choice of the most substantive contributions. At the same time, we also had the ambition that the text can be employed as a textbook, for students who want to master the skill of doing the actual computations.

Above all we wanted to provide a text that is an optimally user-friendly access point to the subject, also for those readers who are not in a position to make the big investments of effort required in order to acquire the full holographist's skill set, but who are eager to get a well-informed impression of the big picture.

To accommodate these conflicting requirements, we have chosen to structure the text in a layered fashion, employing the time-tested use of *boxes*. The main text is rather descriptive, explaining what goes on conceptually and how the computations work. Equations are used only insofar as they are instrumental in getting the big picture across. We then supplement the main text by employing boxes found at the end of sections, where we present the actual calculations in some detail. For those readers who find even the main text too heavy, we have added in addition dual "rule boxes" summarising the punchlines of the neighbouring main text.

The body of the book is formed by chapters 6–13 focussed on the application of holography to condensed matter physics. We introduce these with several initial chapters of background material. Chapters 2 and 3 are intended as a collection of points from condensed matter physics that are of particular relevance to holography. Although these chapters are aimed in the first instance at string theorists and physicists of other fields, we strongly encourage also condensed matter experts to read them, since our presentation deviates significantly from the standard treatment in textbooks. Chapters 4 and 5 are intended to constitute a tutorial in the AdS/CFT apparatus, geared towards its use in the condensed matter context. We have avoided as much as possible the heavy string-theoretical machinery, and instead focussed on the pragmatic use of the correspondence. A reader equipped with a background acquired from entry-level graduate courses in general relativity, quantum field theory and condensed matter physics should be able to comprehend this book in some depth. One issue we had to consider is that detailed calculations in holography often resort to numerics. The level of these is fortunately not very challenging, and to assist the reader we will make the Mathematica codes for several basic calculations available via the website accompanying this book: **www.cambridge.org/9781107080089**.

To offer the reader a first grip on the storyline of this book, let us use the remainder of this introductory chapter to present a grand vista on the AdS/CMT landscape.

1.2 The AdS/CFT correspondence: unifying the theories of physics

The story that we will tell started in the mid 1990s. These years found the string-theory community in a euphoric state since it had become clear that there was much more to string theory than had previously been realised. The culmination of this second string revolution was the discovery in 1997 by the young theorist Juan Maldacena of what has become known as the "AdS/CFT correspondence" [1]. All along, string theory had been propelled by the insight that somehow general relativity is part of the quantum theory of relativistic strings, and that it therefore carried the inherent promise that it would eventually reveal the theory of quantum gravity. AdS/CFT was a great leap forward in this regard.

Quantum field theory (QFT) and general relativity (GR) are the two grand theories of physics, but their mutual relationship is complicated, if not even antagonistic. Maldacena's discovery, however, connected these two pillars in a way that nobody had foreseen. He showed that in a special limit these theories can be two sides of the same coin! The two sides of this unification of quantum physics with general relativity are in a mathematical sense as opposite as can be. This refers to the meaning of "duality" in the title of this book. GR and QFT are in a *dual* relation with each other in the sense of the particle–wave duality of quantum mechanics. Particles and waves are opposites in the sense that they are related by Fourier transformation. But at the same time the particle and wave representations form a wholeness revealing what quantum mechanics is. Depending on the question one asks, either the particle or the wave description is the better viewpoint. In the same sense GR and QFT "merge in their oppositeness", albeit the resulting "wholeness" is much richer than quantum mechanics: in a sense it seems to encapsulate all theories of physics. Soon after Maldacena's discovery Gubser, Klebanov, Polyakov [2] and independently Witten [3] (GKPW) came up with a set of tight and general mathematical rules demonstrating precisely how one can *quantitatively* relate results in one description to the other side. The unveiling of this dictionary launched an enormous research effort: thousands of papers revolving around checks and double checks of the correspondence, applying the new viewpoint to pernicious open problems and diversifying it to a variety of physics subjects were published. This book is dedicated to perhaps the least anticipated success of this headlong dash of exploration: the application to condensed matter physics.

Despite all the progress the correspondence is still shrouded in mystery. Metaphorically it is like the oracle of Greek mythology: upon throwing questions at the correspondence, it delivers answers that make sense, but it is far from clear why it works so well. This is the prevalent moral of the developments we will describe in this book – particularly in the condensed matter context there is a real potential to challenge and mobilise experimental physics to check whether some of the most enigmatic answers of this oracle are actually correct. The mystery of AdS/CFT is rooted precisely in the *quantum* gravity side. In full generality, AdS/CFT relates stringy *quantum* gravity to certain quantum field theories. Stringy quantum gravity remains very poorly understood. However, in a special limit the stringy quantum gravity side reduces to the solid ground of classical general relativity. This limit has a corresponding incarnation in the dual quantum field theory. It requires that the field theory contains matrix-valued fields of rank N and the limit means that one considers the system both in the regime of the "large $N \to \infty$" limit and at "strong coupling". A typical example is a very strongly coupled $SU(N)$ Yang–Mills theory in the limit of a large number of colours N. This sounds rather remote from the real world of condensed matter physics. To make matters even worse, to have full mathematical control in its string theory origin one also has to take a supersymmetric theory. Fortunately, in trying to apply the lessons of AdS/CFT more broadly, supersymmetry appears to be less crucial. The large-N limit is the serious mathematical obstacle to consider. Our main goal will be to study field-theoretical problems that cannot be handled with existing field-theoretical techniques. Within the large-N limit one can now use this AdS/CFT dictionary to call upon the great mathematical quality of Einstein's theory of general relativity to arrive at solutions that with the help of the dictionary can be translated back into field-theoretical answers. But the large-N limit implies that this is trustworthy only on departing from an extremely *symmetric* physics at high energy, which is very different from the mundane "chemistry" of e.g. electrons in solids at the Ångström scale. One would therefore like to lower the N-fold symmetry all the way to the puny symmetries governing the interacting electrons. In principle this is also possible in AdS/CFT. In practice, however, upon doing so the quantum-gravity hell breaks loose in the gravitational dual and, despite a very intense effort, we are still pretty much in the dark.

How can AdS/CFT still deliver? Remarkably, in spite of the large-N obstacle, the oracle seems to deliver answers to questions that do not depend sensitively on these matters. The string theorists refer to the relevant context by invoking "UV-independence". This is coincident with the notion of "strong emergence" of the condensed matter physicists. Both are about the theme that the "whole is more than the sum of the parts", which is familiar to all physicists, where we are interested in the circumstances under which the wholeness has such a strong logic of its own that the detailed nature of the parts is no longer of relevance. It is the independence of macroscopic phenomena from the details of the microscopic physics. This started seriously with Boltzmann's formulation of statistical physics in the nineteenth century for the description of classical matter. It triumphs in the form of an understanding of the solid, fluid and gas phases of simple thermal matter, including the microscopic origins of the phenomenological elasticity and Navier–Stokes theories. This was generalised to the zero-temperature "quantum" realms in condensed matter physics in the form of Landau-style order-parameter theories describing superfluids and superconductors, as well as the Fermi-liquid theory. The next great step was taken by the Wilsonian renormalisation-group revolution in the 1970s, merging together the methodological underpinning of the description of the critical state realised at continuous phase transitions and the fundamental quantum field theories of high-energy physics.

The correspondence builds further on this. Its "magical" power lies in its capacity to express the mathematical structure of the "strongly emergent" theories of matter through the very different geometrical structure of GR. We will unfold this story step by step in chapters 6–12, following closely the historical development, up to the state of the art involving predictions for states of matter that are ruled by macroscopic quantum entanglement.

1.3 AdS/CFT, the geometrisation of the renormalisation group and the quantum critical state

What is actually the meaning of the abbreviations AdS and CFT, and especially the adjective "holographic" in the title of the book? This is all tied together with a mathematical relation that is seen by many as the most beautiful and stunning of all "emergence physics–general relativity" relations. It can be written as an "equation"

$$\mathrm{RG} = \mathrm{GR}. \tag{1.1}$$

Here RG is short for the renormalisation group and GR is of course general relativity. The renormalisation group refers to the property of field theory that by integrating out short-distance degrees of freedom one induces a flow describing how the theory changes as one lowers the scale to longer and longer wavelength. This is enumerated in terms of differential equations expressing the running of coupling constants. What AdS/CFT miraculously tells us is that this scaling "direction" turns into an extra *geometrical dimension* in the gravitational dual. The scaling flow in the field theory is now encoded in the purely geometrical properties of this higher-dimensional gravitational space-time, which in turn is governed by solutions of the Einstein equations. When the field theory lives in a $(d + 1)$-dimensional space-time, the corresponding bulk has $d + 2$ dimensions; this extra dimension is often called the "radial direction". This is metaphorically like a hologram: one has a two-dimensional photographic plate with interference fringes (the field theory) and by shining through a laser beam (the dictionary of the AdS/CFT correspondence) a three-dimensional (3D) image (the bulk) is constructed. The miracle that makes this work is the quantum-gravitational "holographic principle" originating in black-hole physics (see section 4.1). It insists that the counting of degrees of freedom in a gravitational theory behaves similarly to a quantum field theory in one dimension fewer. This is why the members of the large family of AdS/CFT-type correspondences are called "holographic dualities": in the present context this metaphor acquires an extra appeal since we will see that the physics in the gravitational "bulk" is quite recognisable (the 3D images) while the field-theory side is quite abstract and counterintuitive (the interference patterns on the plate).

How precisely do we stitch these different worlds together? It is now the moment to explain the abbreviation. It stands for the particular configuration in which the duality between QFT and GR was first realised. The first part AdS stands for anti-de Sitter space. This is the space-time on the general-relativity side of the correspondence. This is a solution to Einstein's theory characterised by a negative cosmological constant. Geometrically it is the Lorentzian higher-dimensional generalisation of a hyperboloid. The properties of hyperbolic space are famously represented by Escher in a series of drawings, tessellating this space with fishes, devils and reptiles. Formally it appears that such hyperbolic spaces are infinitely large, but here one of the marvels of relativity comes into play. This is not true for light-like propagation. They reach the edge of space-time in a finite time. This means that one has to supplement the gravity theory with specific boundary conditions and boundary information. Since the boundary is naturally of one dimension fewer than the "bulk", one can now imagine that the boundary is correlated with the space-time where the field theory lives, while its RG flow is associated with the extra radial direction moving from this boundary towards the centre of AdS: the "deep interior". This intuitive viewpoint is correct and will serve as a great support in understanding the quantitative dictionary between the two sides of the correspondence. The second abbreviation, CFT, stands for "conformal field theory". In the first explicit realisations of the correspondence the quantum theories were always of a very special type. They were conformally invariant. This means that "everything stays the same under arbitrary scale transformations that preserve angles". This is the category of the theories which are of great interest to contemporary condensed matter physics. Conformally invariant theories explain the universal behaviour at second-order phase transitions. Here, however, these conformal field theories arise naturally as zero-temperature relativistic field theories. They are "quantum critical theories" describing the universal physics near a zero-temperature "quantum phase transition" controlled by another external parameter. Now precisely the notion of quantum criticality appears to play a central role in all the big puzzles of strongly correlated matter revealed by experiment, most notably the strange metals found in high-T_c superconductors and other exotic materials. All along, quantum criticality will be a central motive linking the holographic-duality activities to the laboratory floor.

These are still only glimpses of the beautiful relationship between quantum field theories and general relativity. But, using the pieces just revealed, anti-de Sitter spaces, conformal field theories and the quantum critical state, and the extra direction as somehow encoding the RG flow, we can already give a remarkably intuitive description of the dictionary relating the two sides.

Let us start from the field-theory side of the correspondence, and take a relativistic conformal field theory as our starting point. The natural description of such a CFT is that it lives in a flat, $(d+1)$-dimensional non-dynamical Minkowski space-time, i.e. quantities that respect special relativity are expressed in terms of four-vectors/four-tensors with inner products defined by the metric

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + d\mathbf{x}^2. \tag{1.2}$$

Here $\mu, \nu = 0$ is the time direction, and $\mu, \nu = 1, \ldots, d$ are spatial directions. In addition to manifest invariance under global Lorentz transformations, Minkowski space-time is invariant under time and spatial translations. Correspondingly one has conservation of energy and momentum.

A generic field theory is also subject to renormalisation. Following Wilson, we can integrate out short-distance degrees of freedom consistently, provided that we change the coupling constants g_i of the theory under scale transformations according to differential equations that are local in the RG scale u,

$$u\frac{\partial g_i(u)}{\partial u} = \beta(g_j(u)). \tag{1.3}$$

Right at a critical point, however, the beta functions vanish by definition, $\beta = 0$, and the physics becomes scale-invariant. The combined space and time scale transformation $x^\mu \to \lambda x^\mu$ is now also a symmetry. For a relativistic Lorentz-invariant theory, scale invariance together with unitarity is conjectured to imply invariance under the full set of conformal transformations (see e.g. [4] for a review), i.e. all transformations that preserve angles but not necessarily lengths. These include so-called special conformal transformations that combine with scale and Lorentz transformations to form the group $SO(d+1, 2)$.

Picture now a generic field theory evaluated at different values of the renormalisation-group scale r, and put them in a sequence (Fig. 1.1). If we consider the label r of each theory as a continuous variable, we get a new $(d+2)$-dimensional "space-time" that has one extra dimension that parametrises the RG flow. On the field-theory side, i.e. for each field-theory slice, the RG scale r is of course a non-geometrical entity that lets us know how the field theory behaves as we change the scale. The essence of AdS/CFT is that the *full family* of theories has an alternative dual geometric description in terms of a *real* $(d+2)$-dimensional space-time. To reflect the scaling properties of the underlying field theory, this space-time cannot be flat but has to have a curved shape, exactly as is familiar from Einstein's theory of general relativity. In such curved space-times, distances are measured with the help of the local metric $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$. Remarkably, if the field theory we started from was indeed a very special field theory with conformal and scale invariance, we can deduce what the form of the metric should be, solely from the symmetries of a conformal field theory. Generically any specific metric tensor $g_{\mu\nu}(x)$ also corresponds to a particular "gauge choice" associated with a preferred coordinate frame. However, if the space-time has a true physical symmetry, then the metric should respect this symmetry irrespective of the coordinate choice. Such a symmetry is called an isometry, to distinguish it from the choice-of-coordinate-frame symmetries (general covariance, or diffeomorphism) that are the basis of GR. Since the field theory we wish to describe is conformal and invariant under scale transformations, we must insist that its holographic gravitational dual is so too, if the two sides of the coin are to match. We must therefore

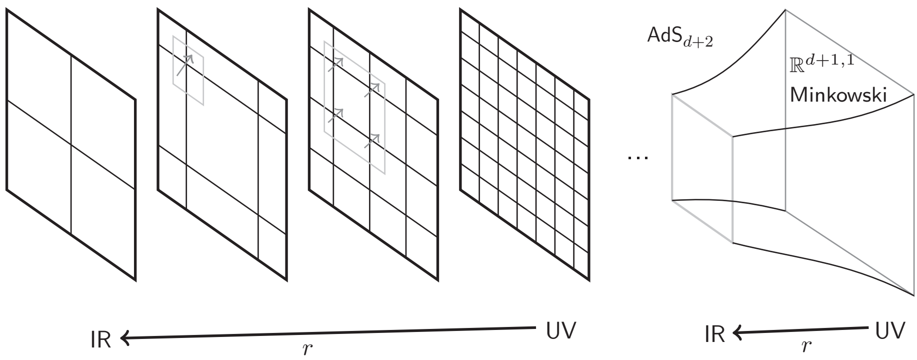

Figure 1.1 Consider the set of copies of a field theory generated by successive coarse-graining steps. Geometrically this set naturally groups together to form a space with one extra spatial direction. The essence of AdS/CFT is that for a conformal field theory (CFT) this grouping can be made mathematically exact. The built-up space-time is an anti-de Sitter space (AdS) in which the extra dimension r in the bulk is naturally interpreted as the renormalisation-group scale in the boundary field theory. Figure source [5].

find a metric in such a way that the total $(d+2)$-dimensional space-time has the scaling symmetry $x^\mu \to \lambda x^\mu$ as an isometry. The "stacking" picture of our field theory, which is Lorentz- and translation-invariant, already tells us that the metric must be of the form

$$ds^2 = f(r)\eta_{\mu\nu}\, dx^\mu\, dx^\nu + g(r)dr^2. \tag{1.4}$$

If we then use the coordinate freedom to set r equal to the energy scale of the field theory, i.e. under scale transformations r should scale as $r \to r/\lambda$, the unique invariant $(d+2)$-dimensional metric is

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}\, dx^\mu\, dx^\nu + \frac{L^2}{r^2}\, dr^2. \tag{1.5}$$

This is precisely the metric of $(d+2)$-dimensional *anti-de Sitter space*. We now see the explicit connection between AdS and CFT. A more detailed study of the AdS metric reveals not only that its *full* set of isometries is Lorentz plus scaling but also that it actually forms the group $SO(d+1,2)$. This matches exactly to the conformal angle-preserving symmetry of a $(d+1)$-dimensional CFT, including the special conformal transformations.

Let's inspect this geometry more closely. The anti-de Sitter AdS$_{d+2}$ geometry is thus a family of copies of Minkowski spaces parametrised by the "radial coordinate" r, whose size is seen to shrink when r decreases from the UV of the field theory at $r \to \infty$ to the IR at $r \to 0$. The "UV" region where $r \to \infty$ is the earlier-mentioned boundary of the space-time, which light-like objects can reach

Table 1.1 *Notations used in this book*

Label	**Physical meaning**
d	the space dimension of the field theory
$d+1$	the space-time dimension of the field theory
$d+2$	the space-time dimension of the gravity theory
L	AdS radius
r	AdS "energy" radial coordinate, $r=0$ ($r_{\rm h}$) is the interior/horizon and $r=\infty$ is the boundary
z	AdS "length" radial coordinate, $z=\infty$ ($z_{\rm h}$) is the interior/horizon and $z=0$ is the boundary
$x_i, \quad i=0,\ldots,d$	time/spatial coordinate of the field theory, and coordinates of AdS transverse to the radial direction

in a finite time. One therefore needs to supply boundary conditions in addition to the dynamical laws, and this will play a very important role throughout. The "IR" region of the field theory is the deep interior of AdS. The free parameter L with the dimension of length is called the "AdS radius", and its meaning for the field theory will become clear later. One can also make the extra holographic dimension an RG "length" by changing coordinates to $z = L^2/r$, for which the metric reads

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2\right). \tag{1.6}$$

We shall use the notations introduced here throughout. For convenience, we summarise them in Table 1.1.

This simple geometrical picture (see Fig. 1.1) already teaches us that the field theory has an intricate relationship with a non-trivial Einstein space-time. The fact that we took a critical scale-invariant theory as the starting point imposed strong restrictions on the emergent geometry of the $(d+2)$-dimensional space-time. A critical theory is very special, however. If we now deform the conformal field theory such that we create a non-trivial flow to a new IR, it is clear that the geometrical structure must change, i.e. the space-time must be able to respond to the physics. The simplest dynamical gravitational theory that fulfils the Landau criteria of satisfying the symmetry requirements (in this case general covariance) with a minimal number of derivatives, and that also has the AdS space-time as a solution, is the Einstein–Hilbert action with an added negative cosmological constant,

$$S = \frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\Big[R - 2\Lambda + \cdots\Big], \tag{1.7}$$

where $g = \det g_{\mu\nu}$, R is the Ricci scalar obtained from the metric, and G is the gravitational coupling constant, while $-2\Lambda = d(d+1)/L^2$ is a negative