Convex Optimization of Power Systems

Optimization is ubiquitous in power system engineering. Drawing on powerful, modern tools from convex optimization, this rigorous exposition introduces essential techniques for formulating linear, second-order cone, and semidefinite programming approximations to the canonical optimal power flow problem, which lies at the heart of many different power system optimizations.

Convex models in each optimization class are then developed in parallel for a variety of practical applications such as unit commitment, generation and transmission planning, and nodal pricing. Presenting classical approximations and modern convex relaxations side-by-side, and a selection of problems and worked examples, this book is an invaluable resource for students and researchers from industry and academia in power systems, optimization, and control.

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Convex Optimization of Power Systems

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For Elodie
# Contents

*Preface*  
Preface page xi  

*Acknowledgments*  
Acknowledgments page xii  

*Notation*  
Notation page xiii  

## 1 Introduction  
1.1 Recent history  
1.2 Structure and outline  
1.3 On approximations  
References page 4  

## 2 Background  
2.1 Convexity and computational complexity  
2.2 Optimization classes  
  2.2.1 Linear and quadratic programming  
  2.2.2 Cone programming  
  2.2.3 Quadratically constrained programming  
  2.2.4 Mixed-integer programming  
  2.2.5 Algorithmic maturity  
2.3 Relaxations  
  2.3.1 Lift-and-project  
  2.3.2 *Detour:* graph theory  
  2.3.3 *Preview:* How to use a relaxation  
2.4 Classical optimization versus metaheuristics  
2.5 Power system modeling  
  2.5.1 Voltage, current, and power in steady-state  
  2.5.2 Balanced three-phase operation  
  2.5.3 Generator and load modeling  
  2.5.4 The per unit system  
2.6 Summary  
References page 39  

## 3 Optimal power flow  
3.1 Basic formulation  
  3.1.1 Nonlinear programming approaches  
References page 44
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>Linear approximations in voltage-polar coordinates</td>
<td>48</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Linearized power flow</td>
<td>48</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Decoupled power flow</td>
<td>49</td>
</tr>
<tr>
<td>3.2.3</td>
<td>Network flow</td>
<td>50</td>
</tr>
<tr>
<td>3.3</td>
<td>Relaxations</td>
<td>51</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Exactness in radial networks</td>
<td>55</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Real coordinate systems</td>
<td>58</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Branch flow models</td>
<td>62</td>
</tr>
<tr>
<td>3.3.4</td>
<td>Further discussion</td>
<td>65</td>
</tr>
<tr>
<td>3.4</td>
<td>Load flow</td>
<td>67</td>
</tr>
<tr>
<td>3.4.1</td>
<td>Exact load flow</td>
<td>67</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Linearized load flow</td>
<td>69</td>
</tr>
<tr>
<td>3.5</td>
<td>Extensions</td>
<td>70</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Direct current networks</td>
<td>70</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Reactive power capability curves</td>
<td>71</td>
</tr>
<tr>
<td>3.5.3</td>
<td>Nonconvex generator cost curves</td>
<td>72</td>
</tr>
<tr>
<td>3.5.4</td>
<td>Polyhedral relaxation of the second-order cone</td>
<td>74</td>
</tr>
<tr>
<td>3.6</td>
<td>Summary</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>References</td>
<td>77</td>
</tr>
<tr>
<td>4</td>
<td>System operation</td>
<td>81</td>
</tr>
<tr>
<td>4.1</td>
<td>Multi-period optimal power flow</td>
<td>81</td>
</tr>
<tr>
<td>4.1.1</td>
<td>Ramp constraints</td>
<td>83</td>
</tr>
<tr>
<td>4.1.2</td>
<td>Energy storage and inventory control</td>
<td>84</td>
</tr>
<tr>
<td>4.1.3</td>
<td>Implementation via model predictive control</td>
<td>89</td>
</tr>
<tr>
<td>4.2</td>
<td>Stability and control</td>
<td>90</td>
</tr>
<tr>
<td>4.2.1</td>
<td>The swing equation</td>
<td>91</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Linear quadratic regulation</td>
<td>94</td>
</tr>
<tr>
<td>4.3</td>
<td>Unit commitment</td>
<td>96</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Objective</td>
<td>97</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Constraints</td>
<td>98</td>
</tr>
<tr>
<td>4.4</td>
<td>Reconfiguration</td>
<td>101</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Radiality constraints</td>
<td>102</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Power flow and objectives</td>
<td>103</td>
</tr>
<tr>
<td>4.4.3</td>
<td>Transmission switching</td>
<td>106</td>
</tr>
<tr>
<td>4.5</td>
<td>Summary</td>
<td>107</td>
</tr>
<tr>
<td></td>
<td>References</td>
<td>108</td>
</tr>
<tr>
<td>5</td>
<td>Infrastructure planning</td>
<td>112</td>
</tr>
<tr>
<td>5.1</td>
<td>Nodal placement and sizing</td>
<td>113</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Problem types and greedy algorithms</td>
<td>114</td>
</tr>
<tr>
<td>5.1.2</td>
<td>Power sources</td>
<td>116</td>
</tr>
<tr>
<td>5.1.3</td>
<td>Multiple scenarios</td>
<td>118</td>
</tr>
<tr>
<td>5.1.4</td>
<td>Energy storage</td>
<td>120</td>
</tr>
</tbody>
</table>
# Contents

5.2 Transmission expansion 120
5.2.1 Basic approach 121
5.2.2 Linearized models 123
5.2.3 Branch flow approximation 125
5.2.4 Relaxations 126
5.2.5 Feasibility issues 128
5.3 Summary 129
References 130

6 Economics 132
6.1 Background 133
6.1.1 Lagrangian duality 133
6.1.2 Pricing and the welfare theorems 139
6.1.3 Game theory 141
6.2 Electricity markets 144
6.2.1 Nodal pricing 148
6.2.2 Multi-period and dynamic pricing 157
6.2.3 Transmission cost allocation 160
6.2.4 Pricing under nonconvexity 166
6.3 Market power 167
6.3.1 Supply function equilibrium 170
6.3.2 Complementarity models 172
6.3.3 Capacitated price competition 173
6.4 Summary 176
References 178

7 Future directions 184
7.1 Uncertainty modeling 184
7.1.1 Stochastic programming 184
7.1.2 Robust optimization 185
7.2 Decentralization and distributed optimization 186
7.3 More game theory 188
7.3.1 Dynamic games 188
7.3.2 Mechanism design 189
References 190

Index 193
Preface

The application of optimization to power systems has become so common that it deserves treatment as a distinct subject. The abundance of optimization problems in power systems can give the impression of diversity, but in truth most are merely layers on a common core: the steady-state description of power flow in a network. In this book, many of the most prominent examples of optimization in power systems are unified under this perspective.

As suggested by the title, this book focuses exclusively on convex frameworks, which by reputation are phenomenally powerful but often too restrictive for realistic, non-convex power system models. In Chapter 3, the application of classical and recent mathematical techniques yields a rich spectrum of convex power flow approximations ranging from high tractability and low accuracy to slightly reduced tractability and high accuracy. The remaining chapters explore problems in power system operation, planning, and economics, each consisting of details layered on top of the convex power flow approximations. Because all formulations can be solved using standard software packages, only models are presented, which is a departure from most books on power systems. It is a major perk of convex optimization that the user often does not need to program an algorithm to proceed.

I should comment that this book is not an up-to-date exposition of power system applications or optimization theory and that, inevitably, many important topics in both fields have been omitted. My intention has rather been to bridge modern convex optimization and power systems in a rigorous manner. While I have attempted to be mathematically self-contained, the pace assumes an advanced undergraduate level of mathematical exposure (linear algebra, calculus, and some probability) as well as familiarity with power systems and optimization. This book could be used in a course on power system optimization or as a mathematical supplement to a course in power system design, operation, or economics. It is my hope that it will also prove useful to researchers in power systems with an interest in optimization and vice versa, and to industry practitioners seeking firm foundations for their optimization applications.
I started this book in the fall of 2011. Most of the present content I learned under the guidance of Franz Hover during my graduate school years at the Massachusetts Institute of Technology and of Duncan Callaway, Kameshwar Poolla, and Pravin Variaya during my postdoctoral studies at the University of California, Berkeley. Certainly, without the open-minded environments they created, this project would have never been attempted. I must also thank a number of colleagues whom I’ve benefitted from regular discussions with: Eilyan Bitar, Brendan Englot, Reza Iravani, Deepa Kundur, Johanna Mathieu, Daniel Muenz, Ashutosh Nayyar, Andy Packard, Matias Negrete-Pincetic, Anand Subramanian, and many others who have made power systems, control, and optimization such pleasant fields to work in. Finally, Julie Lancashire and Sarah Marsh at Cambridge University Press have made the final stages of this book a highly enjoyable process.
Notation

AC Alternating current
DC Direct current
LP Linear programming
QP Quadratic programming
SOC(P) Second-order cone (programming)
SD(P) Semidefinite (programming)
(C)QCP (Convex) quadratically constrained programming
MI Mixed integer
NLP Nonlinear programming
KKT Karush-Kuhn-Tucker (conditions)
PNE Pure strategy Nash equilibrium
MNE Mixed strategy Nash equilibrium

\[
i = \sqrt{-1}
\]

\[
\mathbb{R} \quad \text{The set of real numbers}
\]

\[
\mathbb{C} \quad \text{The set of complex numbers}
\]

\[
\mathbb{Z} \quad \text{The set of integers}
\]

\[
x_i \quad \text{The } i^{th} \text{ entry of the vector } x
\]

\[
x^k \quad \text{The } k^{th} \text{ version of the quantity } x, \text{ typically corresponding to the } k^{th} \text{ scenario or time period}
\]

\[
\text{Re } x \quad \text{The real part of } x
\]

\[
\text{Im } x \quad \text{The imaginary part of } x
\]

\[
|x| \quad \text{The absolute value of } x
\]

\[
||x|| \quad \text{The two-norm of } x, \sqrt{\sum_i x_i^2}
\]

\[
X_{ij} \quad \text{The entry at the } i^{th} \text{ row and } j^{th} \text{ column of the matrix } X
\]

\[
X^T \quad \text{The transpose of } X
\]

\[
X^* \quad \text{The Hermitian transpose of } X. \text{ When } X \text{ is scalar, the complex conjugate.}
\]

\[
X \succeq 0 \quad \text{The matrix } X \text{ is positive semidefinite.}
\]

\[
\text{rank } X \quad \text{The rank of } X
\]

\[
\text{tr } X \quad \text{The trace of } X, \sum_i X_{ii}
\]

\[
\det X \quad \text{The determinant of } X
\]

\[
\nabla \quad \text{The gradient operator}
\]
To condense exposition, this book employs somewhat relaxed indexing notation. Because there is little risk of ambiguity, $i$ will often be used simultaneously as the imaginary unit and as an index, for example $iq_i$ would be $\sqrt{-1}$ times the $i^{th}$ entry of $q$. In most cases, constraint indexing will not be explicitly declared; for example,

$$g_i(x) \leq 0$$

is implicitly enforced over $i = 1, \ldots, n$, which is almost always the set of nodes in the network. Similarly, the sum

$$\sum_{ij} x_{ij}$$

is over all relevant node pairs $ij$, which are usually those connected by lines. Indexing is denoted explicitly when it is not over a standard set, such as when summing over a subset of nodes.

This book makes extensive use of feasible sets as organizational tools. Given a collection of constraints $g_j(x) \leq 0$, the corresponding feasible set is

$$\{x \mid g_j(x) \leq 0\},$$

i.e., the set of points for which every constraint is satisfied.