The Surprising Mathematics of Longest Increasing Subsequences

In a surprising sequence of developments, the longest increasing subsequence problem, originally mentioned as merely a curious example in a 1961 paper, has proven to have deep connections to many seemingly unrelated topics in mathematics, such as random matrices and interacting particle systems. The detailed and playful study of these connections makes this book suitable as a starting point for a wider exploration of elegant mathematical ideas that are of interest to every mathematician and to many computer scientists, physicists, and statisticians.

Among the specific topics covered are the Vershik–Kerov–Logan–Shepp limit shape theorem, the Baik–Deift–Johansson theorem, the Tracy–Widom distribution, and the corner growth process. This exciting body of work, encompassing important advances in probability and combinatorics over the last 40 years, is made accessible to a general graduate-level audience for the first time in a highly polished presentation.

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Cambridge University Press
978-1-107-07583-2 - The Surprising Mathematics of Longest Increasing Subsequences
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32 Avenue of the Americas, New York, NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781107428829

First published 2015

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data
Romik, Dan, 1976– author.
The surprising mathematics of longest increasing subsequences / Dan Romik.
pages cm
Includes bibliographical references and index.
ISBN 978-1-107-07583-2 (hardback) – ISBN 978-1-107-42882-9 (paperback)
1. Combinatorial analysis. 2. Probabilities. I. Title.
QA164.R66 2014
511′.6–dc23 2014023514

ISBN 978-1-107-07583-2 Hardback
ISBN 978-1-107-42882-9 Paperback

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Preface

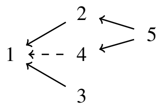
> "Good mathematics has an air of economy and an element of surprise."
>
> – Ian Stewart, *From Here to Infinity*

As many students of mathematics know, mathematical problems that are simple to state fall into several classes: there are those whose solutions are equally simple; those that seem practically impossible to solve despite their apparent simplicity; those that are solvable but whose solutions nonetheless end up being too complicated to provide much real insight; and finally, there are those rare and magical problems that turn out to have rich solutions that reveal a fascinating and unexpected structure, with surprising connections to other areas that lie well beyond the scope of the original problem. Such problems are hard, but in the most interesting and rewarding kind of way.

The problems that grew out of the study of longest increasing subsequences, which are the subject of this book, belong decidedly in the latter class. As readers will see, starting from an innocent-sounding question about random permutations we will be led on a journey touching on many areas of mathematics: combinatorics, probability, analysis, linear algebra and operator theory, differential equations, special functions, representation theory, and more. Techniques of random matrix theory, a sub-branch of probability theory whose development was originally motivated by problems in nuclear physics, will play a key role. In later chapters, connections to interacting particle systems, which are random processes used to model complicated systems with many interacting elements, will also surface. Thus, in this journey we will have the pleasure of tapping into a rich vein of mathematical knowledge, giving novices and experts alike fruitful avenues for exploration. And although the developments presented in this book are fairly modern, dating from the last 40 years, some of the tools we will need are based on classical 19th century mathematics. The fact that such old mathematics can be repurposed for use in new ways that could never have been imagined by its original discoverers is a delightful demonstration of what the physicist Eugene P. Wigner [146] (and later Hamming [55] and others) once famously described as the "unreasonable effectiveness" of mathematics.

Because the subject matter of this book involves such a diverse range of areas, rather than stick to a traditional textbook format I chose a style of presentation a bit similar in spirit to that of a travel guide. Each chapter is meant to take readers on an exploration of ideas covering a certain mathematical landscape, with the main goal being to prove some deep and difficult result that is the main "tourist attraction" of the subject being covered. Along the way, tools are developed, and sights and points of interest of less immediate importance are pointed out to give context and to inform readers where they might go exploring on their next visit.

Again because of the large number of topics touched upon, I have also made an effort to assume the minimum amount of background, giving quick overviews of relevant concepts, with pointers to more comprehensive literature when the need arises. The book should be accessible to any graduate student whose background includes graduate courses in probability theory and analysis and a modest amount of previous exposure to basic concepts from combinatorics and linear algebra. In a few isolated instances, a bit of patience and willingness to consult outside sources may be required by most readers to understand the finer points of the discussion. The dependencies between chapters are shown in the following diagram:



(Chapter 4 is only minimally dependent on Chapter 1 for some notation and definitions.)

The book is suitable for self-study or can be covered in a class setting in roughly two semester-long courses. Exercises at many levels of difficulty, including research problems of the "do not try this at home" kind, are included at the end of each chapter.

The subjects covered in the different chapters are as follows. Chapter 1 presents the **Ulam–Hammersley problem** of understanding the asymptotic behavior of the maximal length of an increasing subsequence in a uniformly random permutation as the permutation order grows. After developing the necessary tools the chapter culminates in the first solution of the problem by Vershik–Kerov and Logan–Shepp. Chapter 2 covers the beautiful **Baik–Deift–Johansson theorem** and its extension due to Borodin–Okounkov–Olshanski and Johansson – a major refinement of the picture revealed by Vershik–Kerov and Logan–Shepp that ties the problem of longest increasing subsequences to the Tracy–Widom distribution from random matrix theory and to other important concepts like determinantal point processes. Chapter 3 discusses **Erdős–Szekeres permutations**, a class of permutations possessing extremal behavior with respect to their maximal monotone subsequence lengths, which are analyzed by applying and extending the techniques developed in Chapter 1.

Chapters 4 and 5 are devoted to the study of the **corner growth process**, a random walk on Young diagrams that bears an important conceptual resemblance to another process introduced in Chapter 1. In Chapter 4 we prove the well-known limiting shape result of Rost and its extension to the case of corner growth in discrete time. Chapter 5 then develops a new approach to the problem, due to Johansson, that enables proving a much more precise fluctuation result, again involving the Tracy–Widom distribution.

I am grateful to the people and organizations who helped make this book possible. My work was supported by the National Science Foundation under grant DMS-0955584; by grant 228524 from the Simons Foundation; and of course by my excellent employer of the last 5 years, the University of California, Davis. I also received advice, suggestions, error reports, and encouragement from Arvind Ayyer, Eric Brattain-Morrin, Peter Chang, Alexander Coward, Ira Gessel, Geoffrey Grimmett, Indrajit Jana, Donald Knuth, Christian Krattenthaler, Greg Kuperberg, Isaac Lambert, Liron Mor Yosef, Vladimir Pchelin, Yuval Peres, Amir Sarid, Sasha Soshnikov, Perla Sousi, Mike Steele, and Peter Winkler. Ron Peled outdid everyone by sending me so many insightful suggestions for improvement that I had to beg him to stop, and deserves special thanks.

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May 2014