Praise for An Introduction to Stochastic Dynamics

"Jinqiao Duan's book introduces the reader to the actively developing theory of stochastic dynamics through well-chosen examples that provide an overview, useful insights, and intuitive understanding of an often technically complicated topic."

– P. E. Kloeden, Goethe University, Frankfurt am Main

"Randomness is an important component of modeling complex phenomena in biological, chemical, physical, and engineering systems. Based on many years teaching this material, Jinqiao Duan develops a modern approach to the fundamental theory and application of stochastic dynamical systems for applied mathematicians and quantitative engineers and scientists. The highlight is the staged development of invariant stochastic structures that underpin much of our understanding of nonlinear stochastic systems and associated properties such as escape times. The book ranges from classic Brownian motion to noise generated by α-stable Levy flights."

– A. J. Roberts, University of Adelaide

AN INTRODUCTION TO STOCHASTIC DYNAMICS

The mathematical theory of stochastic dynamics has become an important tool in the modeling of uncertainty in many complex biological, physical, and chemical systems and in engineering applications – for example, gene regulation systems, neuronal networks, geophysical flows, climate dynamics, chemical reaction systems, nanocomposites, and communication systems. It is now understood that these systems are often subject to random influences, which can significantly impact their evolution.

This book serves as a concise introductory text on stochastic dynamics for applied mathematicians and scientists. Starting from the knowledge base typical for beginning graduate students in applied mathematics, it introduces the basic tools from probability and analysis and then develops for stochastic systems the properties traditionally calculated for deterministic systems. The book's final chapter opens the door to modeling in non-Gaussian situations, typical of many real-world applications. Rich with examples, illustrations, and exercises with solutions, this book is also ideal for self-study.

JINQIAO DUAN is professor and director of the Laboratory for Stochastic Dynamics at Illinois Institute of Technology. During 2011–13, he also served as professor and associate director of the Institute for Pure and Applied Mathematics (IPAM) at UCLA. An expert in stochastic dynamics, stochastic partial differential equations, and their applications in engineering and science, he has been the managing editor for the journal *Stochastics and Dynamics* for more than a decade. He is also a coauthor of the research monograph *Effective Dynamics of Stochastic Partial Differential Equations* (2014).

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Dedicated to the memory of
my grandparents, Duan Chongxiang and Ye Youxiang
and
my parents, Duan Jianhua and Chen Yuying

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Preface

Dynamical systems are often subject to random influences, such as external fluctuations, internal agitation, fluctuating initial conditions, and uncertain parameters. In building mathematical models for these systems, some less-known, less-well-understood, or less-well-observed processes (e.g., highly fluctuating fast- or small-scale processes) are ignored because of a lack of knowledge or limitations in our analytical skills and computational capability. This ignorance also contributes to uncertainty in mathematical models of complex dynamical systems. However, uncertainty or randomness may have a delicate and profound impact on the overall evolution of complex dynamical systems. Indeed, there is clear recognition of the importance of taking randomness into account when modeling complex phenomena in biological, chemical, physical, and other systems.

Stochastic differential equations are usually appropriate models for randomly influenced systems. Although the theoretical foundation for stochastic differential equations has been provided by stochastic calculus, better understanding the dynamical behaviors of these equations is desirable.

Who Is This Book For?

There is growing interest in stochastic dynamics in the applied mathematics community. This book is written primarily for applied mathematicians who may not have the necessary background to go directly to advanced reference books or the research literature in stochastic dynamics. My goal is to provide an introduction to basic techniques for understanding solutions of stochastic differential equations, from analytical, deterministic, computational, and structural perspectives. In deterministic dynamical systems, invariant manifolds and other invariant structures provide global information for dynamical evolution. For stochastic dynamical systems, in addition to these invariant structures, certain computable quantities, such as the mean exit time and escape probability (reminiscent of quantities like "eigenvalues" and

"Poincaré index" in deterministic dynamics and "entropy" in statistical physics), also offer insights into global dynamics under uncertainty. The mean exit time and escape probability are computed by solving deterministic, local or nonlocal, partial differential equations. Thus, I treat them as deterministic tools for understanding stochastic dynamics. It is my hope that this book will help the reader in accessing advanced monographs and the research literature in stochastic dynamics.

What Does This Book Do?

A large part of the material in this book is based on my lecture notes for the graduate course Stochastic Dynamics that I have taught many times since 1997. Among the students who have taken this course, about two-thirds are from applied mathematics, and the remaining one-third are from departments such as physics, computer science, bioengineering, mechanical engineering, electrical engineering, and chemical engineering. I would like to thank those graduate students for helpful feedback and for solutions to some exercises. For this group of graduate students, selection of topics and choice of presentation style are necessary. Thus, some interesting topics are not included. The choice of topics is personal but is influenced by my teaching these graduate students, who have basic knowledge in differential equations, dynamical systems, probability, and numerical analysis. Some materials are adopted from my recent research with collaborators; these include the most probable phase portraits in Chapter 5 and random invariant manifolds in Chapter 6, together with mean exit time, escape probability, and nonlocal Fokker-Planck equations for systems with non-Gaussian Lévy noise in Chapter 7.

I have tried to strike a balance between mathematical precision and accessibility for the readers of this book. For example, some proofs are presented, whereas some are outlined and others direct to the references. Some definitions are presented in separate paragraphs starting with "Definition," but many others are introduced less formally as they occur in the body of the text. As far as possible, I have tried to make connections between new concepts in stochastic dynamics and old concepts in deterministic dynamics.

After some motivating examples (Chapter 1), background in analysis and probability (Chapter 2), a mathematical model for white noise (Chapter 3), and a crash course in stochastic differential equations (Chapter 4), I focus on three topics:

- **Quantities that carry stochastic dynamical information (Chapter 5):** This includes moments, probability densities, most probable phase portraits, mean exit time, and escape probability.
- **Structures that build stochastic dynamics (Chapter 6):** This includes the multiplicative ergodic theorem and Hartman-Grobman theorem for linearized stochastic systems and invariant manifolds for nonlinear stochastic systems.

- **Non-Gaussian stochastic dynamics (Chapter 7):** This is an introduction to systems driven by non-Gaussian, α-stable Lévy motion.

This book is full of examples, together with many figures. There are separate Matlab simulation sections in Chapters 2–4, whereas in Chapters 5 and 7, numerical simulations are included in various sections. Although Chapter 6 contains no numerical simulations for its nature, it has examples and problems that require detailed derivations or calculations by hand. At the end of each chapter are homework problems, including some numerical simulation problems; Matlab is sufficient for this purpose. Most of these problems have been tested in the classroom. Hints or solutions to most problems are provided at the end of the book.

A section with an asterisk may be skipped on a first reading.

Some additional references are provided in the "Further Readings" section, for more advanced readers.

What Prerequisites Are Assumed?

For the reader, it is desirable to have basic knowledge of dynamical systems, such as the material contained in

- Chapters 1–2 of *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* by J. Guckenheimer and P. Holmes
- Chapters 1–2 of *Introduction to Applied Nonlinear Dynamical Systems and Chaos* by S. Wiggins
- Chapters 1–2 of *Differential Equations and Dynamical Systems* by L. Perko
- Chapters 1–3 of *Nonlinear Dynamics and Chaos* by S. H. Strogatz

Ideally, it is also desirable to have elementary knowledge of stochastic differential equations, such as

- Chapters 1–6 of *Stochastic Differential Equations* by L. Arnold
- Chapters 1–5 of *An Introduction to Stochastic Differential Equations* by L. C. Evans
- Chapters 1–5 of *Stochastic Differential Equations* by B. Oksendal
- Chapters 1–3 of *Stochastic Methods* by C. Gardiner

Realizing that some readers may not be familiar with stochastic differential equations, I review this topic in Chapter 4.

Acknowledgments

I would like to thank Philip Holmes for suggesting that I write this book back in 2004, when we were taking an academic tour in China. Steve Wiggins has also

encouraged me to publish this book. I am especially grateful to Ludwig Arnold, who has always inspired and encouraged my learning and research in stochastic dynamics. I appreciate Bernt Oksendal's encouragement and comments. I have benefited from many years of productive collaboration and interaction with many colleagues, especially Peter Bates, Peter Baxendale, Dirk Blomker, Tomas Caraballo, Michael Cranston, Hans Crauel, Manfred Denker, David Elworthy, Franco Flandoli, Hongjun Gao, Martin Hairer, Peter Imkeller, Peter E. Kloeden, Kening Lu, Navaratnam Sri Namachchivaya, Anthony Roberts, Michael Scheutzow, Bjorn Schmalfuss, Richard Sowers, Xu Sun, Yong Xu, and Huaizhong Zhao. My interest in non-Gaussian stochastic dynamics started with a joint paper with D. Schertzer, M. Larcheveque, V. V. Yanovsky, and S. Lovejoy in 2000 and was further inspired and enhanced by Peter Imkeller and Ilya Pavlyukevich during my sabbatical leave at Humboldt University in Berlin in 2006.

Ludwig Arnold proofread this entire book and provided invaluable comments and suggestions. Han Crauel, Peter Imkeller, Peter Kloeden, Jicheng Liu, Guangying Lu, Mark Lytell, Bjorn Schmalfuss, Renming Song, Xiangjun Wang, and Jiang-Lun Wu proofread parts of this book, and their comments and corrections helped improve the book in various ways.

I thank Jia-an Yan for helpful discussions about topics in Chapter 4. I am very grateful to Zhen-Qing Chen, Xiaofan Li, Huijie Qiao, Renming Song, Xiangjun Wang, and Jiang-Lun Wu for helpful discussions about topics in Chapter 7. My former and current graduate students, especially Xiaopeng Chen, Hongbo Fu, Ting Gao, Zhongkai Guo, Tao Jiang, Xingye Kan, Mark Lytell, Jian Ren, Jiarui Yang, and Yayun Zheng, have helped with generating figures, proofreading some chapters, and providing solutions to some problems.

I would also like to acknowledge the National Science Foundation for its many years of generous support of my research. A part of this book was written while I was at the Institute for Pure and Applied Mathematics (IPAM), Los Angeles, during 2011–13. Diana Gillooly at Cambridge University Press has provided valuable professional help for the completion of this book.

My wife, Yan Xiong, and my children, Victor and Jessica, are constant sources of inspiration and happiness. Their love and understanding made this book possible.

Jinqiao Duan
Chicago, April 2014

Notation

$\triangleq$ is defined to be

$|x|$ absolute value of $x \in \mathbb{R}^1$

$\|x\|$ Euclidean norm of $x \in \mathbb{R}^n$

$a \wedge b \triangleq \min\{a, b\}$

$a \vee b \triangleq \max\{a, b\}$

$a^+ \triangleq \max\{a, 0\}$

$a^- \triangleq \max\{-a, 0\}$

B_t Brownian motion

$\mathcal{B}(\mathbb{R}^n)$ Borel σ-field of $\mathbb{R}^n$

$\mathcal{B}(S)$ Borel σ-field of state space S

$C(\mathbb{R}^n)$ space of continuous functions on $\mathbb{R}^n$

$C_0(\mathbb{R}^n)$ space of continuous functions on $\mathbb{R}^n$ that have compact support

$C^k(\mathbb{R}^n)$ space of continuous functions on $\mathbb{R}^n$ that have up to kth-order continuous derivatives

$C_0^k(\mathbb{R}^n)$ space of continuous functions on $\mathbb{R}^n$ that (1) have up to kth-order continuous derivatives and (2) have compact support

$C^\infty(\mathbb{R}^n)$ space of continuous functions on $\mathbb{R}^n$ that have derivatives of all orders

$C_0^\infty(\mathbb{R}^n)$ space of continuous functions on $\mathbb{R}^n$ that (1) have derivatives of all orders and (2) have compact support

$C^\alpha(D)$ space of functions that are locally Hölder continuous in D with exponent α

$C^\alpha(\bar{D})$ space of functions that are uniformly Hölder continuous in D with exponent α

$C^{k,\alpha}(D)$ space of continuous functions in D whose kth-order derivatives are locally Hölder continuous in D with exponent α

$\delta(\xi)$ Dirac delta function

$\mathbb{E}$ expectation

$\mathbb{E}^x$ expectation with respect to the probability measure $\mathbb{P}^x$ induced by a solution process starting at x

$F_X(x)$ distribution function of the random variable X

$\mathcal{F}^X$ or $\sigma(X)$ σ-field generated by the random variable X

$\mathcal{F}^{X_t} = \sigma(X_s, s \in \mathbb{R})$ σ-field generated by a stochastic process X_t; it is the smallest σ-field with which X_t is measurable for every t

$\mathcal{F}^{\xi}$ σ-field generated by the stochastic process ξ_t

$\mathcal{F}_t^B \triangleq \sigma(B_s : s \leq t)$ filtration generated by Brownian motion B_t

$\mathcal{F}_\infty \triangleq \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$

$\mathcal{F}_{t^+} \triangleq \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$

$\mathcal{F}_{t^-} \triangleq \sigma(\bigcup_{s<t} \mathcal{F}_s)$

$\mathcal{F}_t^X \triangleq \sigma(X_s : 0 \leq s \leq t)$ filtration generated by a stochastic process X_t

$\mathcal{F}_{-\infty}^t \triangleq \sigma(\cup_{s\leq t}\mathcal{F}_s^t)$ also denoted as $\bigvee_{s\leq t} \mathcal{F}_s^t$

$\mathcal{F}_s^\infty \triangleq \sigma(\cup_{t\geq s}\mathcal{F}_s^t)$ also denoted as $\bigvee_{t\geq s} \mathcal{F}_s^t$

$H(f)$ Hessian matrix of a scalar function $f : \mathbb{R}^n \to \mathbb{R}^1$

$H(\xi)$ Heaviside function

$H^k(D)$ Sobolev space

$H_0^k(D)$ Sobolev space of functions with compact support

$\|\cdot\|_k$ Sobolev norm in $H^k(D)$ or $H_0^k(D)$

lim in m.s. convergence in mean square, i.e., convergence in $L^2(\Omega)$

l^p space of infinite sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty |x_i|^p < \infty$

$L^2(\mathbb{R}^n)$ space of square-integrable functions defined on $\mathbb{R}^n$

$L^p(\mathbb{R}^n)$ space of p-integrable functions defined on $\mathbb{R}^n$, with $p \geq 1$

$L^p(D)$ space of p-integrable functions defined on a domain $D \subset \mathbb{R}^n$, with $p \geq 1$

$L^2(\Omega)$ or $L^2(\Omega, \mathbb{R}^n)$ space of random variables, taking values in Euclidean space $\mathbb{R}^n$, with finite variance

$L^2(\Omega)$ or $L^2(\Omega, H)$ space of random variables, taking values in Hilbert space H, with finite variance

$L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^n)$ $\{X : \mathbb{E}|X|^p < \infty\}$ for $p \geq 1$

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$L_t(\omega)$ Lévy motion

$L_t^\alpha(\omega)$ α-stable Lévy motion

$\mathbb{N}$ set of the natural numbers

$\mathcal{N}(\mu, \sigma^2)$ normal (or Gaussian) distribution with mean μ and variance σ^2

$\nu(dy)$ Lévy jump measure

$\mathbb{P}$ probability measure

$\mathbb{P}(A)$ or $\mathbb{P}\{A\}$ probability of an event A

P^X distribution measure induced by the random variable X

$\mathbb{P}^x$ probability measure induced by a solution process starting at x

$P(\lambda)$ Poisson distribution with parameter $\lambda > 0$

$\mathbb{R}$ two-sided time axis

$\mathbb{R}^+$ one-sided time set $\{t : t \geq 0\}$

$\mathbb{R}^1$ one-dimensional Euclidean space

$\mathbb{R}^n$ n-dimensional Euclidean space

$\sigma(X)$ or $\mathcal{F}^X$ σ-field generated by the random variable X; it is the smallest σ-field with which X is measurable

$\mathrm{Supp}(f) \triangleq$ closure of $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ support of function f

$\mathrm{Tr}(A)$ trace of A

$U(a, b)$ uniform distribution on the interval $[a, b]$

$\bigvee_{s \leq t} \mathcal{F}_s^t \triangleq \sigma(\cup_{s \leq t} \mathcal{F}_s^t)$ also denoted as $\mathcal{F}_{-\infty}^t$

$\bigvee_{t \geq s} \mathcal{F}_s^t \triangleq \sigma(\cup_{t \geq s} \mathcal{F}_s^t)$ also denoted as $\mathcal{F}_s^{\infty}$