1

Introduction

Noisy fluctuations are abundant in complex systems. In some cases, noise is not negligible, whereas in some other situations, noise could even be beneficial. It is desirable to have a better understanding of the impact of noise on dynamical evolution of complex systems. In other words, it becomes crucial to take randomness into account in mathematical modeling of complex phenomena under uncertainty.

In 1908, Langevin devised a stochastic differential equation for the motion of Brownian particles in a fluid, under random impacts of surrounding fluid molecules. This stochastic differential equation, although important for understanding Brownian motion, went largely unnoticed in the mathematical community until after stochastic calculus emerged in the late 1940s. Introductory books on stochastic differential equations (SDEs) include [8, 88, 213].

The goal for this book is to examine and present select dynamical systems concepts, tools, and methods for understanding solutions of SDEs. To this end, we also need basic information about deterministic dynamical systems modeled by ordinary differential equations (ODEs), as presented in the first couple of chapters in one of the references [110, 290].

In this introductory chapter, we present a few examples of deterministic and stochastic dynamical systems, then briefly outline the contents of this book.

1.1 Examples of Deterministic Dynamical Systems

We recall a few examples of deterministic dynamical systems, where short time-scale forcing and nonlinearity can affect dynamics in a profound way.

Example 1.1 *A double-well system.* Consider a one-dimensional dynamical system $\dot{x} = x - x^3$. It has three equilibrium states, -1, 0, and 1, at which the vector field

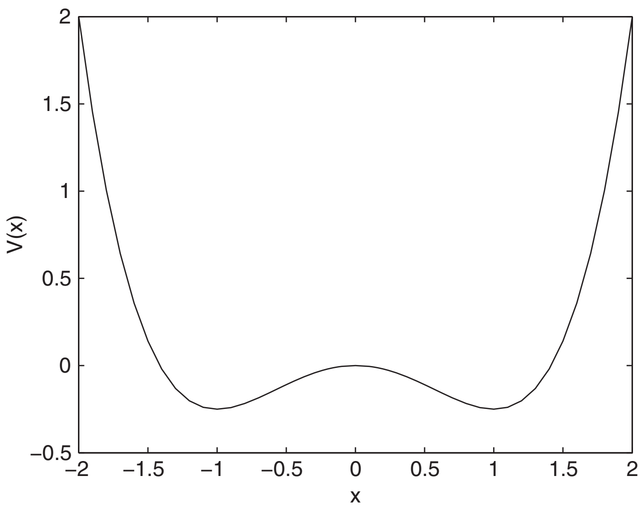

Figure 1.1 Plot of $V(x) \triangleq -\frac{1}{2}x^2 + \frac{1}{4}x^4$.

$x - x^3$ is zero. Observe that

$$\dot{x} = x - x^3 = x(1 - x^2) = \begin{cases} < 0, & -1 < x < 0 \quad \text{or} \quad 1 < x < \infty, \\ = 0, & x = -1, 0, 1, \\ > 0, & -\infty < x < -1 \quad \text{or} \quad 0 < x < 1. \end{cases}$$

Note that $\dot{x} = x - x^3 \triangleq -\frac{dV(x)}{dx}$, where the potential function $V(x) \triangleq -\frac{1}{2}x^2 + \frac{1}{4}x^4$ has two minimal values (sometimes called "wells"); see Figure 1.1.

A solution curve, or orbit, or trajectory, starting with $x(0) = x_0$ in $(-1, 0)$, decreases in time (because $\dot{x} < 0$ on this interval) and approaches the equilibrium state -1 as $t \to +\infty$, whereas an orbit starting with $x(0) = x_0$ in $(-\infty, -1)$ increases in time (because $\dot{x} > 0$ on this interval) and approaches the equilibrium state -1 as $t \to +\infty$. Thus the equilibrium point $\{-1\}$ is a stable equilibrium state and is an *attractor*, that is, it attracts nearby orbits. Likewise $\{1\}$ is also an attractor. But the equilibrium state $\{0\}$ is unstable and is called an *repeller*. See Figure 1.2 for a few representative solutions curves.

An orbit starting near one equilibrium state $\{-1\}$ cannot go anywhere near the other equilibrium state $\{1\}$, and vice versa. There is no transition between these two stable states.

If we only look at the solution curves in the state space, $\mathbb{R}^1$, where state x lives, we get a state portrait, or as it is often called, a phase portrait.

Figure 1.3 shows the phase portrait for this double-well system.

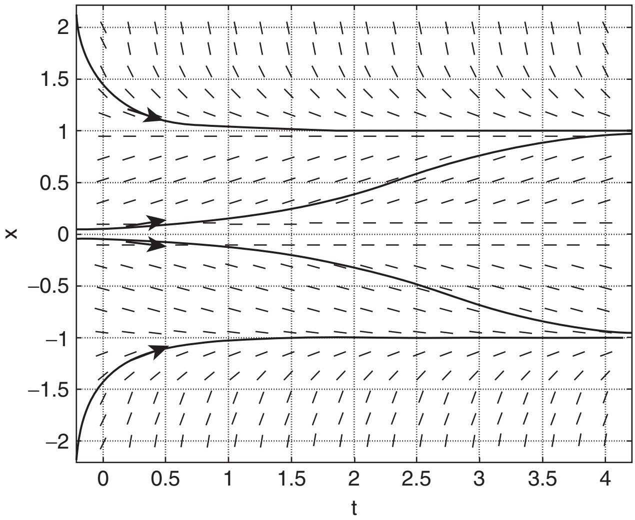

Figure 1.2 Solution curves for $\dot{x} = x - x^3$.

Example 1.2 *High-frequency (or short time-scale) forcing.* Consider a simple one-dimensional nonlinear system with time-periodic forcing with frequency ω:

$$\dot{x} = -x + x^3 + \varepsilon \sin(\omega t), \quad x(0) = 0.5. \tag{1.1}$$

Solution curves with frequency $\omega = 2$ and $\omega = 10$ are shown in Figures 1.4 and 1.5, respectively. The difference between low- and high-frequency forcing is visible.

Example 1.3 *Small nonlinearity leads to fundamental change in dynamics.* Consider a harmonic oscillator (a spring-mass system) of mass m and spring constant k, under damping that is proportional to the cubic of velocity: $m\ddot{x} = -kx - \varepsilon\dot{x}^3$, where ε is a positive constant. For simplicity, we take m, k both equal to 1. This can also be achieved by rescaling the time. Thus,

$$\ddot{x} = -x - \varepsilon\dot{x}^3 \tag{1.2}$$

or equivalently,

$$\dot{x} = y, \tag{1.3}$$

$$\dot{y} = -x - \varepsilon y^3, \tag{1.4}$$

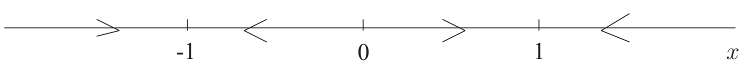

Figure 1.3 Phase portrait for $\dot{x} = x - x^3$.

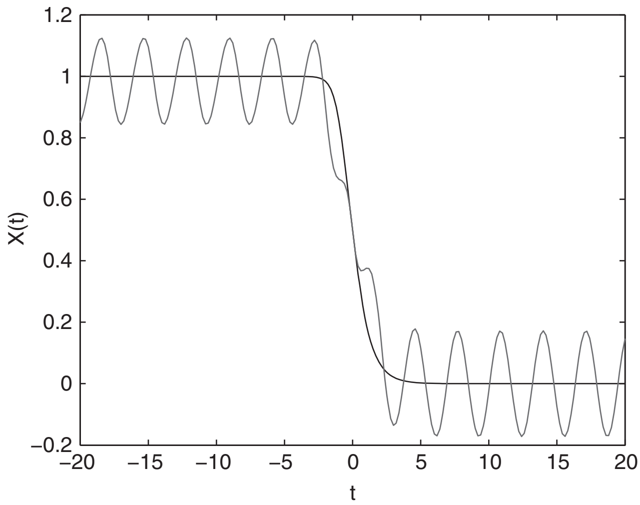

Figure 1.4 Solutions of $\dot{x} = -x + x^3 + \varepsilon \sin(\omega t)$, $x(0) = 0.5$ with frequency $\omega = 2$: $\varepsilon = 0$ (no "oscillations" or black line) and $\varepsilon = 0.35$ (with "oscillations" or gray line).

where x is the displacement and y is the velocity of the oscillator. The equilibrium state is (0, 0).

Without damping ($\varepsilon = 0$), the model equations become

$$\dot{x} = y, \tag{1.5}$$

$$\dot{y} = -x. \tag{1.6}$$

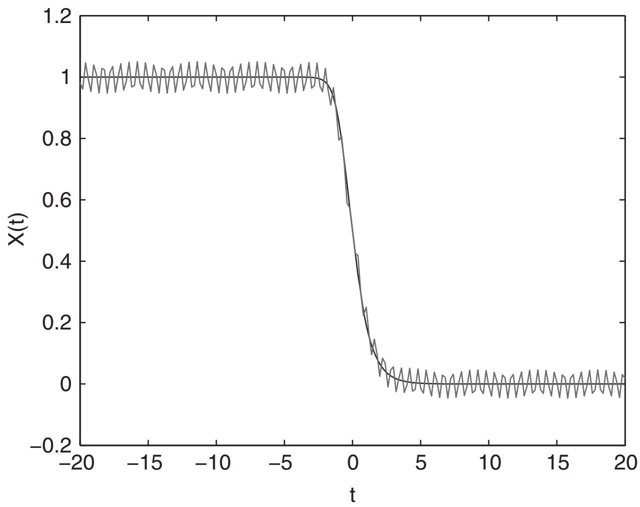

Figure 1.5 Solutions of $\dot{x} = -x + x^3 + \varepsilon \sin(\omega t)$, $x(0) = 0.5$ with frequency $\omega = 10$: $\varepsilon = 0$ (no "oscillations" or black line) and $\varepsilon = 0.35$ (with "oscillations" or gray line).

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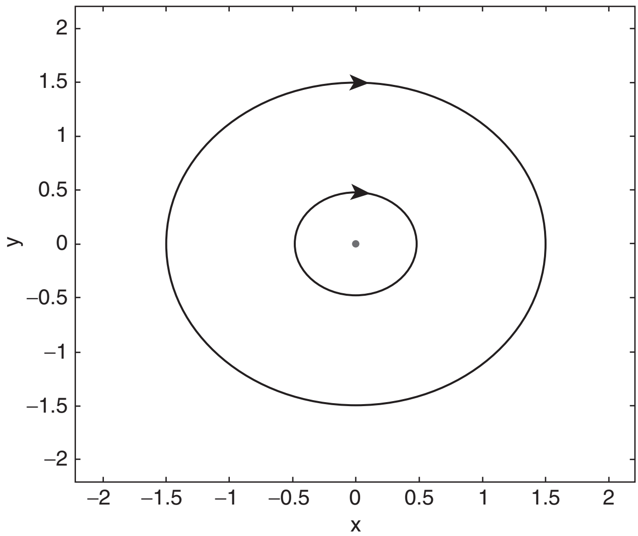

Figure 1.6 Phase portrait for harmonic oscillator $\dot{x} = y$, $\dot{y} = -x - \varepsilon y^3$: $\varepsilon = 0$ (no damping).

Dividing these two equations, we obtain

$$\frac{dy}{dx} = -\frac{x}{y}$$

or

$$xdx + ydy = 0.$$

Integrating this equation, we see that the solution curves $(x(t), y(t))$ satisfy the conservation of energy

$$x^2 + y^2 = c \tag{1.7}$$

for an arbitrary (nonnegative) constant of integration, c. Thus, the solution curves are circles; see Figure 1.6.

In the case of damping, that is, when $\varepsilon > 0$, the energy is not conserved,

$$\frac{d}{dt}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = 2xy + 2y(-x - \varepsilon y^3) = -2\varepsilon y^4 < 0,$$

at all points, except the equilibrium point $(0, 0)$. Thus, all orbits approach the equilibrium point $(0, 0)$ as $t \to \infty$, no matter how small the damping coefficient ε is, as shown in Figure 1.7. Comparing Figure 1.6 and Figure 1.7, we see that the dynamics, with or without damping, are drastically different.

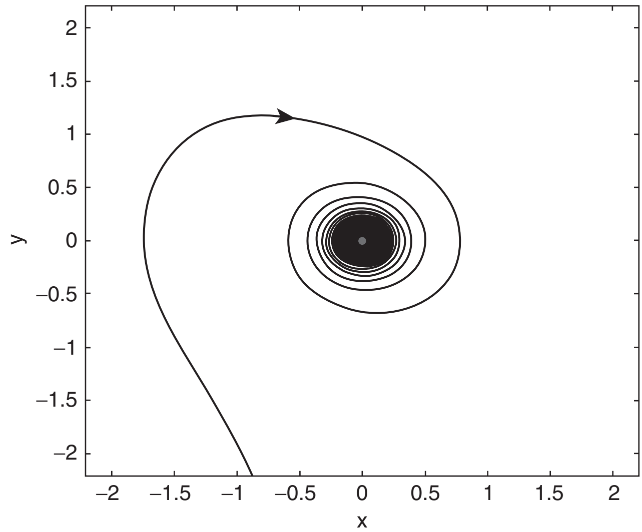

Figure 1.7 Phase portrait for damped harmonic oscillator $\dot{x} = y$, $\dot{y} = -x - \varepsilon y^3$: $\varepsilon = 0.5$.

Example 1.4 *Simple pendulum.* Consider a simple pendulum of mass m and length l: $\ddot{x} = -\frac{g}{l}\sin x$ where x is the angular displacement from the vertical downward (equilibrium) position. Note that m does not appear in the equation. Introducing a time change $\tau \triangleq \sqrt{\frac{g}{l}}\, t$, but still denoting $\frac{dx}{d\tau}$ by $\dot{x}$, we have

$$\ddot{x} = -\sin x \tag{1.8}$$

or, equivalently,

$$\dot{x} = y, \tag{1.9}$$

$$\dot{y} = -\sin x, \tag{1.10}$$

where y is the angular velocity of the pendulum. The equilibrium states are $(0, 0)$ and $(\pm n\pi, 0)$, for $n = 1, 2, \ldots$.

Dividing these two equations, we obtain

$$\frac{dy}{dx} = -\frac{\sin x}{y}$$

or, equivalently,

$$\sin x dx + y dy = 0.$$

This leads to the conservation of energy

$$\frac{1}{2}y^2 - \cos x = c$$

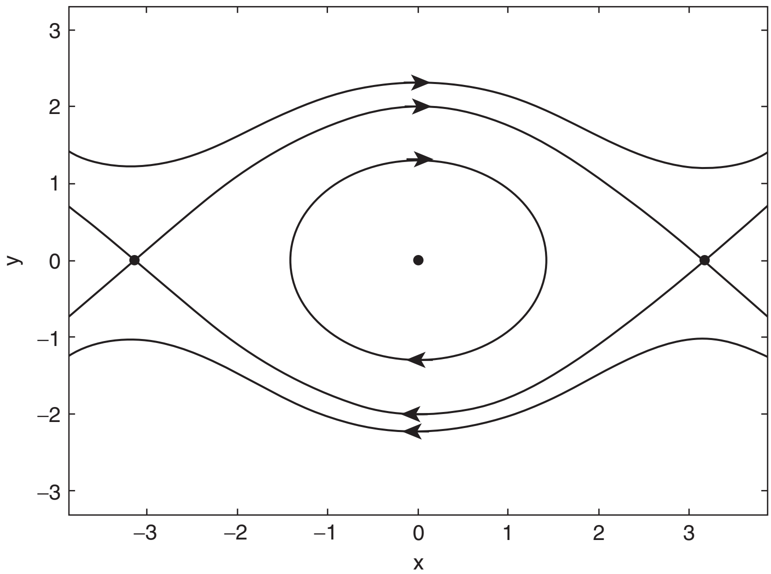

Figure 1.8 Phase portrait for simple pendulum: $\dot{x} = y, \;\; \dot{y} = -\sin x$.

for an arbitrary constant c. See Figure 1.8 for the phase portrait of this simple pendulum system.

Now consider the simple pendulum under damping, $\ddot{x} = -\sin x - \varepsilon \dot{x}$, with $\varepsilon > 0$. This equation is rewritten as

$$\dot{x} = y, \tag{1.11}$$

$$\dot{y} = -\sin x - \varepsilon y. \tag{1.12}$$

In this case, the energy is not conserved,

$$\frac{d}{dt}\left(\frac{1}{2}y^2 - \cos x\right) = y\dot{y} + (\sin x)\dot{x} = y(-\sin x - \varepsilon y) + (\sin x)y = -\varepsilon y^2 < 0,$$

for all (x, y), except the equilibrium points. As was the case of Example 1.3, the dynamics with damping is very different, no matter how small the parameter ε is. See Figure 1.9 for the phase portrait of this damped simple pendulum.

Example 1.5 *A coupled system: Small change in one part affects the others dramatically.* Consider a coupled system

$$\dot{x} = 0.001x - xy, \tag{1.13}$$

$$\dot{y} = -6y + \varepsilon x^2, \tag{1.14}$$

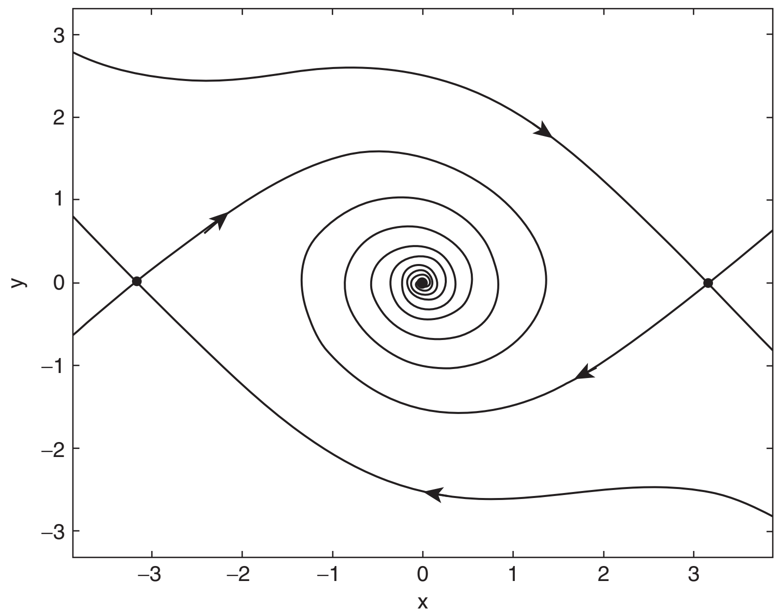

Figure 1.9 Phase portrait for simple pendulum with damping: $\dot{x} = y$, $\dot{y} = -\sin x - \varepsilon y$ with $\varepsilon = 0.25$.

where ε is a small real parameter. A small change in the y-part affects the dynamics dramatically: when $\varepsilon = 0$, the system has an unbounded attracting set (i.e., the whole x-axis), whereas for $\varepsilon = 0.01$, the system has a bounded attracting set (in fact, an inertial manifold). Compare Figures 1.10 and 1.11.

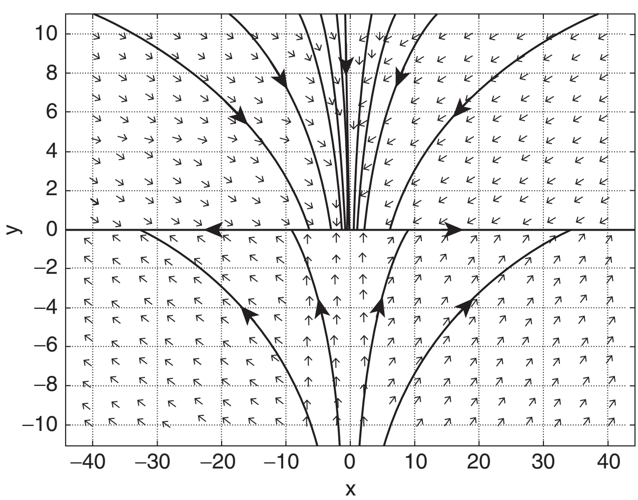

Figure 1.10 Phase portrait for $\dot{x} = 0.001x - xy$, $\dot{y} = -6y$.

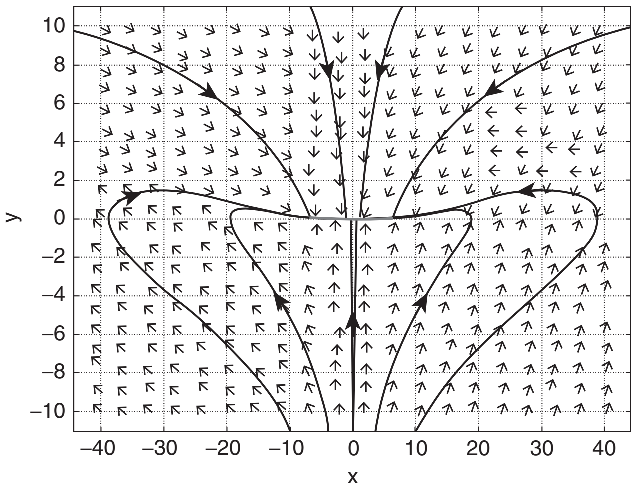

Figure 1.11 Phase portrait for $\dot{x} = 0.001x - xy$, $\dot{y} = -6y + 0.01x^2$: the global attractor is clearly seen (gray curve within $-7 < x < 7$ and near the x-axis).

1.2 Examples of Stochastic Dynamical Systems

Let us look at a few examples of stochastic differential equations arising in the modeling of complex phenomena from various disciplines.

Example 1.6 *Motion of "particles" subject to a random force, in physics, biophysics, and geophysics*. A particle moving in one dimension x, subject to a driving force $K(x)$, a friction force $-\gamma\dot{x}$ with a nonnegative parameter γ, and a random force $\xi(t)$, is described by [280, 179]:

$$\ddot{x} + \gamma\dot{x} = K(x) + \xi(t). \tag{1.15}$$

This includes the model equation of motion for a simple pendulum under a random force,

$$\ddot{x} = \sin(x) + \xi(t), \tag{1.16}$$

where x is now the angular displacement.

A similar stochastic differential equation, still in one dimension, arises to model a mechanism (socalled ergodic pumping) that drives biomolecular conformation changes [189, 190]. This equation in two or three dimensions is a model for the motion of geophysical fluid particles under random influences [40, 229, 108].

Example 1.7 *A tumor growth system with immunization.* In the absence of an immune reaction, tumor evolution is thought to follow a growth law that can be approximated by a logistic function. With the tumor-immune interactions taken into account, a dynamical evolution model for a tumor growth system with immunization was recently proposed [37]:

$$\dot{x} = ax - bx^2 - \frac{\beta x^2}{1 + x^2} + \xi(t), \tag{1.17}$$

where x is the density of tumor cells, a is the deterministic growth rate, b is the decay rate, and β is the strength of the immunization. The term $\xi(t)$ represents the uncertain impact on the growth rate of tumor tissue by environmental factors, such as the supply of nutrients, the immunological state of the host, chemical agents, temperature, and radiation.

Example 1.8 *Mirror symmetry breaking and restoration in biomolecules under random fluctuations.* Molecules that are mirror images of each other are called enantiomers. However, this mirror or chiral symmetry is broken in all biological systems. When this symmetry breaking occurred and whether it may be restored are unanswered questions. The chiral structures include proteins, almost always as the left-handed enantiomers (L), and DNA, RNA polymers, and sugars with chiral building blocks composed by right-handed (D) monocarbohydrates. To study this chiral symmetry breaking and restoration, a stochastic model is introduced [124] to quantify the dynamical evolution of the enantiomeric excess $x = ([L] - [D])/([L] + [D])$, which is the order parameter for mirror symmetry breaking, and the net chiral matter $y = [L] + [D]$,

$$\dot{x} = -2k_1 Ax/y + \frac{1}{2}(k_3 - k_{-2})xy(1 - x^2) + \xi(t), \tag{1.18}$$

$$\dot{y} = 2k_1 A - y^2 \left[k_{-2} + \frac{1}{2}(k_3 - k_{-2})(1 - x^2) \right] + (k_2 A - k_{-1})y, \tag{1.19}$$

where $\xi(t)$ is a noise process due to internal and external fluctuations of the chemical system, A is the achiral reactant (kept at constant concentration), and $k_{\pm 1}$, $k_{\pm 2}$ together with k_3 are various reaction, amplification, or inhibition rates.

Example 1.9 *Dynamics of two inhibitory neural units.* Brown and Holmes [42] considered a stochastic dynamical model for two mutually inhibiting, leaky, neural units characterized by state variables x_j, subject to external stimuli ρ_j, additive noises modeled by independent Brownian motions (to be defined in Chapter 3) $B_{j(t)}$, and "priming biases" $i_0 + b_j$, including an overall level i_0 and separate unit biases b_j. Each unit inhibits the other via an activation function $f(x; g, b) = 1/[1 + \exp(-g(x - b))]$ with gain g that achieves half level at $x = b$.