FLUID DYNAMICS

This book presents a focused, readable account of the principal physical and mathematical ideas at the heart of fluid dynamics. Graduate students in engineering, applied mathematics, and physics who are taking their first graduate course in fluids will find this book invaluable in providing the background in physics and mathematics necessary to pursue advanced study.

The book includes a detailed derivation of the Navier-Stokes and energy equations, followed by many examples of their use in studying the dynamics of fluid flows. Modern tensor analysis is used to simplify the mathematical derivations, thus allowing a clearer view of the physics. Peter S. Bernard also covers the motivation behind many fundamental concepts such as Bernoulli’s equation and the stream function.

Many exercises are designed with a view toward using MATLAB or its equivalent to simplify and extend the analysis of fluid motion, including developing flow simulations based on techniques described in the book.

Peter S. Bernard has 35 years’ experience teaching graduate-level fluid mechanics at the University of Maryland. He is a Fellow of the American Physical Society and Associate Fellow of the American Institute of Aeronautics and Astronautics. In addition to his many research articles devoted to the physics and computation of turbulent flow, he is the coauthor of the highly regarded volume *Turbulent Flow: Analysis, Measurement, and Prediction.*
Fluid Dynamics

Peter S. Bernard
University of Maryland
To my wife, Susan Bradshaw Sullivan
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Preface

This book is inspired by a graduate-level course in fluid dynamics that I have taught at the University of Maryland for many years. The typical student taking this course, which is the starting point for graduate studies in fluid mechanics, has had one undergraduate course on fluids and a limited exposure to vector and tensor analysis. Consequently, the goal of this book is to provide a background in the physics and mathematics of fluid mechanics necessary for the pursuit of advanced studies and research at the graduate level. It is my experience that an effective route to these objectives is via a synthesis of the best features of two very excellent books, namely, *An Introduction to Fluid Dynamics* by George Batchelor, which presents the physics of fluid mechanics with exceptional clarity, and *An Introduction to Continuum Mechanics* by M. E. Gurtin (and now expanded and revised as *The Mechanics and Thermodynamics of Continua* by Gurtin, Fried, and Anand), which demonstrates the advantages of direct tensor notation in simplifying the expression of physical laws. Thus, to a large extent, this book combines the physics of Batchelor with the mathematics of Gurtin. The hope is that, in this way, an environment is created that helps make the subject of fluid dynamics clear, focused, and readily understandable. As a practical matter, this book should serve as an effective stepping-stone for new graduate students to enhance their accessibility to the books by Batchelor and Gurtin as well as those by many others.

Stylistically, this book follows an arc through the material that builds steadily toward the derivation and then application of the Navier-Stokes equations. The sequence of topics is also chosen so as to provide some significant exposure to examples of fluid flow and problem solving, before a relatively long and unavoidable set of chapters that deal in detail with the derivation of the flow equations. Most of what is in this book is covered in a one-semester course at Maryland, and no attempt is made to provide the depth of topics covered by Batchelor or Gurtin nor the comprehensive treatment of the subject matter typically found in other advanced textbooks. After studying this book, it is hoped that students will be well prepared to venture in any number of directions into more specialized and advanced topics in fluid dynamics.

Among the topics in the book, some represent a review of subjects normally encountered in undergraduate fluids courses (e.g., Chapter 9, on fluids at rest). This is intended to keep the book self-contained, to aid in the review of this material and
as a needed introduction to these topics for the occasional applied math or other nonengineering student who has never previously studied fluid mechanics.

The problems at the end of the chapters attempt to reflect the graduate level of the book by pursuing directions that are often somewhat challenging rather than repeating the formulaic engineering problems that are traditional to the undergraduate curriculum. For many of the problems, students are strongly encouraged to take full advantage of high-order computer languages such as MATLAB to help derive relations via symbolic manipulation, to solve algebraic and differential equations, and to calculate and plot numerical results. For example, in the case of MATLAB, facility with using commands such as `diff`, `int`, `solve`, `dsolve`, `subs`, `ode45`, and `bvp4c` greatly reduces the labor necessary to solve many problems in this book. In some cases, without the power of the symbolic solvers, the difficulty in obtaining solutions can be quite formidable if attempted with pencil and paper.

Some of the material in the book is specifically designed to be a launching point for writing computer code (e.g., with MATLAB) that solves interesting flow problems and displays results in the form of animations. Such material includes Sections 18.4 and 18.5, on the discrete vortex method; Section 19.5, on the motion of a sphere and other bodies; and Section 21.2, on the use of the Monte Carlo method for simulating scalar transport in fluid flows. In each of these cases, the numerical simulations can be carried out with a modest investment in programming yet bring to life intriguing aspects of fluid flow.

The author would like to express his great appreciation to Professor Bruce Berger for his many contributions toward improving the quality and clarity of the exposition in this book. I also appreciate the insights of Carl Biagetti of the Space Telescope Science Institute and graduate student Eric Leonard in reading some of the chapters.