1 Introduction

1.1 What Is a Fluid?

What distinguishes fluids from solids? Because we all have an intuitive understanding of the difference, the question is really about precisely identifying the formal differences between them. In this regard, it is useful to consider the different ways that fluids and solids react to applied forces. For example, in the case of a solid, the material offers resistance if we press down on it. If the applied force is not so large as to shatter the solid, then it is clear that the solid is quite capable of resisting the force so as to reach a state of equilibrium. The solid arrives at a state where it ceases to move or deform.

Consider now specifically the case of a gas, as shown in Fig. 1.1. In the figure, a piston is pushing down on the gas, and although initially the gas might compress because of the applied force, it is also able to eventually reach a point for any given applied force where it does not compress further. In other words, the gas is capable of resisting the downward force in the same way as the solid. We may conclude that resistance to a normal force is not a good candidate for framing the distinguishing properties of fluids and solids.

The situation for fluids and solids is different if we consider an applied shear force, as in the experiment indicated in Fig. 1.2. Following the application of a shear force to the top surface of the solid, as shown in Fig. 1.2(a), an equilibrium is reached in which the body has deformed a fixed amount. Alternatively, if the container holds a fluid, as in Fig. 1.2(b), and a shear force is applied to the top lid, the fluid cannot prevent the lid from sliding to the side. This is true no matter how small the applied force may be. This is not to say that the fluid does not offer resistance – it does – but the resistance it offers cannot be enough to create a stationary equilibrium. It is this very different behavior between solids and fluids that distinguishes one from the other in a formal sense. Unlike solids, fluids are unable to prevent the motion caused by a shearing force, no matter how small the shearing force might be.

1.2 Molecular Structure and the Continuum Hypothesis

Fluids and solids behave differently because of the differences in their molecular structures. Whereas solids tend to have a fixed arrangement of molecules, in a

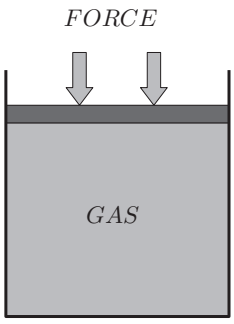
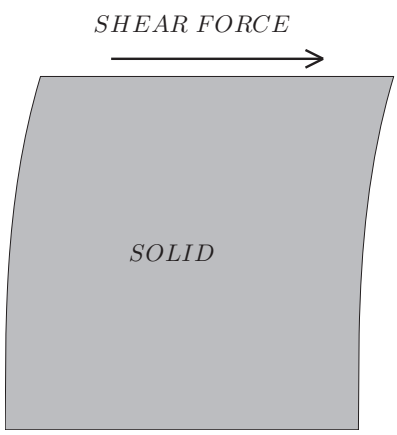


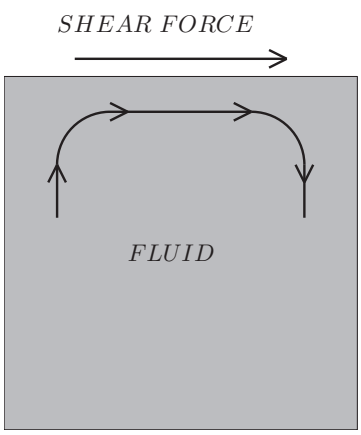
Figure 1.1. A gas can successfully resist an applied normal force to achieve equilibrium.

liquid or gas, molecules are in continuous relative motion. In the case of gases, the molecules generally move at very high velocities. For example, in hydrogen at room temperature, the mean speed of the molecules is approximately 2000 m/s, or faster than 4000 MPH, with much of that motion associated with the internal energy of the gas. The molecules tend to be separated from each other by a distance on the order of the mean free path. As they move, they encounter other molecules in the form of collisions, in fact, having a vast number of collisions per second, in the neighborhood of 10^{10} under typical atmospheric conditions. The collisions are efficient at rapidly spreading information among the molecules about changes to their external conditions. For example, if heat is added at one place in the fluid so that the local molecules at that position move faster, then the energy contained in this motion spreads rapidly to more distant molecules via collisions.

During the compression of a gas, such as occurs in Fig. 1.1, the molecules nearest to the piston react first to the new circumstances by a local increase in pressure and density, a state that the rest of the gas becomes aware of and reacts to through the passage of information by collisions. The news of the change in properties generally travels at the speed of sound, which is on the order of 340 m/s in air.



(a) Solids resist shear forces to achieve equilibrium.



(b) Fluids resist the applied shear but cannot reach a static equilibrium.

Figure 1.2. Response of solids and fluids to a shearing force.

Liquids also can compress under an applied force, although in this case, the molecules are much closer together than in a gas and are better able to resist compression. In fact, the molecules in a liquid are separated from each other by the distance at which the repulsive strong force originating in quantum effects between molecules crosses to a weak attractive electrochemical force. Changing the average distance between molecules in these circumstances requires a large applied force. For a liquid, the mutual jostling of molecules in close proximity owing to the molecular forces provides the means by which there is a rapid flow of information about changed circumstances from one place to another in the fluid.

The subject of fluid mechanics as it is often applied to the study of engineering flows is not concerned directly with the molecular structure underlying liquids and gases. Instead, attention is focused on field quantities, such as the velocity field $\mathbf{u}(\mathbf{x}, t)$ that is defined at all positions $\mathbf{x}$ covering a region of flow at times t contained within a specified interval. In three dimensions, $\mathbf{u}$ has scalar components in a rectangular Cartesian coordinate system that will be denoted here either as $(u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ or $(u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t))$, depending on circumstances. Similarly, the position vector has components $\mathbf{x} = (x_1, x_2, x_3)$, which will also be frequently referred to as (x, y, z). Among the many other fields of interest in fluid mechanics are the density $\rho(\mathbf{x}, t)$, the pressure $p(\mathbf{x}, t)$ with dimension of force/area, and the internal energy/mass $E(\mathbf{x}, t)$. The connection between a field quantity, such as $\mathbf{u}$, ρ, p, E, and the relevant property of the molecules of which the fluid is composed is for the most part conceptional. In other words, it may be assumed that field quantities represent a theoretical average of the respective property held by the molecules within a small sensing volume around the point $\mathbf{x}$ at given time t. The average is expected to be independent of the size of this volume for a range of scales that are small in comparison to the scale at which variations in the flow property might be noticeable in the macroscopic world, say, a fraction of a millimeter. This condition is well met under most circumstances. The precise size of the sensing volume is immaterial, however, because it is not used in deriving the fluid equations, nor does the determination of fluid flow depend on knowing it.

Figure 1.3 shows the result of a model computation of ρ using MATLAB for different sizes of the sensing volume. In the figure, 3 million points are randomly scattered within a unit square, and the average number of points contained within a box of decreasing size is counted and then used to estimate the local density. It is seen that ρ fluctuates rapidly as the box size is varied through small values, indicating that this is a range where the sensing volume is too small. The density estimate is seen to be constant at larger sensing volumes, and this is the value of the density that would be referred to in a field quantity such as $\rho(\mathbf{x}, t)$. For even larger sensing volumes, it can be imagined that the average density will begin to change once again as it becomes affected by measurable (i.e., human scale) gradients in the local fluid properties.

The fact that it is not necessary to have a definite relationship between field quantities and molecular averages is the substance of the *continuum hypothesis*. In effect, it is assumed that the behavior of a real fluid with properties such as $\mathbf{u}$ and ρ can be modeled by the dynamics of an imaginary fluid that is for the most part continuous and lacks any molecular structure. This means that, if at time $t = 0$, one were to measure $\mathbf{u}$ and ρ in a real fluid using an appropriate sensing volume and then

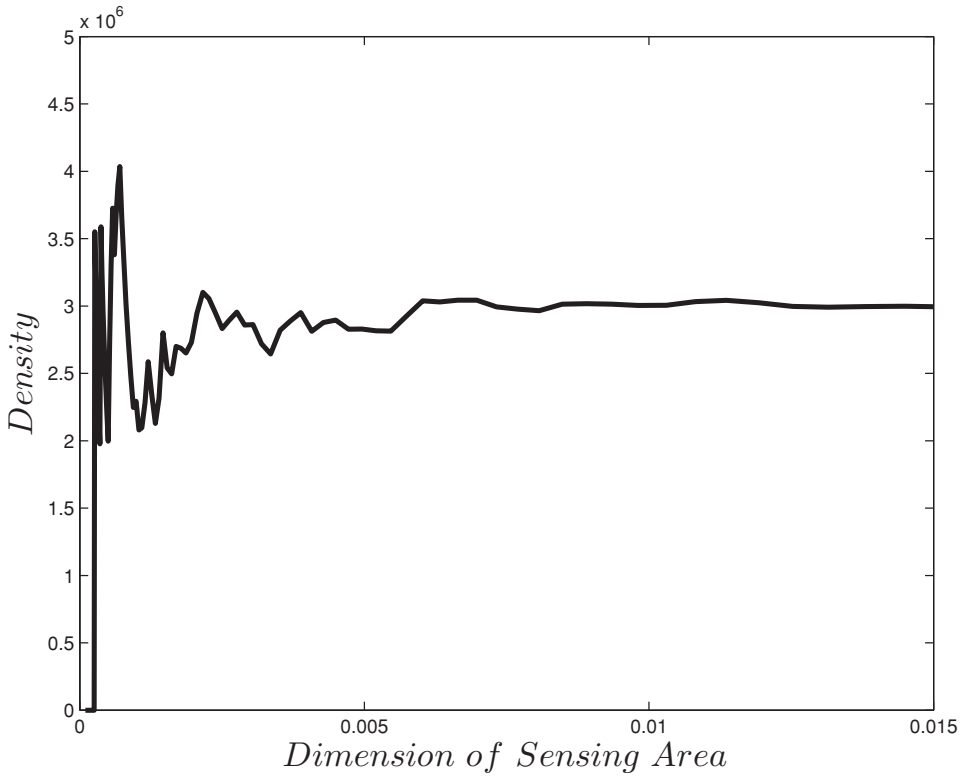


Figure 1.3. Dependence of the density at a point on the sensing area for 3 million points randomly distributed in a unit square.

advance these quantities further in time using laws derived for the continuum fluid, then at subsequent times, $\mathbf{u}$ and ρ will once again be equal to their respective averages computed from the local molecules. The way in which molecular effects do enter into the equations for the continuum variables, as seen later, is via material properties such as the viscosity, μ, and specific heat, c. The magnitude of such variables as well as their functional dependence on quantities such as the temperature vary from one material to the next because of differences in molecular structure and need to be independently assigned values to make quantitative predictions of fluid fields in a continuum model.

1.3 Dilatation and Vorticity

The velocity field $\mathbf{u}(\mathbf{x}, t)$ is fundamental to the description of fluid flow and is often the primary means by which the flow field is understood. Alternatively, two fundamental field quantities formed out of derivatives of $\mathbf{u}$ are seen to play many important roles in characterizing fluid motion. The first is the divergence of the velocity field, $\nabla \cdot \mathbf{u}$, which is referred to as the *dilatation* and is often denoted in this book as $\theta(\mathbf{x}, t)$. In rectangular Cartesian coordinates,

$$\theta = \nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{1.1}$$

where here and henceforth summation from 1 to 3 is implied in repeated indices such as in the expression $\partial u_i/\partial x_i$ appearing in Eq. (1.1). In Section 10.2, it is shown that zero dilatation is precisely the condition that is necessary and sufficient for the flow to be incompressible, that is, for the volume of any particular element of fluid to remain the same as it moves through the flow.

The second fundamental quantity that can be computed from $\mathbf{u}$ is the vorticity field, $\boldsymbol{\omega}(\mathbf{x}, t)$, defined as the curl of the velocity field

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}, \tag{1.2}$$

or, in index notation,

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}, \tag{1.3}$$

where the alternating tensor

$$\epsilon_{ijk} \equiv \begin{cases} 1 & i, j, k \rightarrow \quad \text{cyclic order} \\ -1 & i, j, k \rightarrow \quad \text{anticyclic order,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Cyclic order means that $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, and anticyclic order means that $(i, j, k) = (3, 2, 1), (2, 1, 3), (1, 3, 2)$. A calculation shows that

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \tag{1.5a}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \tag{1.5b}$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{1.5c}$$

In two-dimensional flow, in which case $\mathbf{x} = (x, y)$ and the velocity field is $\mathbf{u} = (u, v)$, according to Eqs. (1.5), the only nonzero component of vorticity is ω_3, that is, $\boldsymbol{\omega} = (0, 0, \omega_3)$, or more simply, $\boldsymbol{\omega} = (0, 0, \omega)$. In this case, it is seen that the vorticity vector is oriented perpendicular to the plane of motion.

A subsequent discussion in Section 12.1 shows that the vorticity may be understood physically as being twice the instantaneous angular velocity of a local fluid element under the hypothetical circumstance that it is instantaneously frozen and allowed to move subsequently as a solid body. Thus, vorticity is generally associated with rotational motion in a fluid, though it is not necessarily true that the fluid is rotating when the vorticity is nonzero. The degree to which $\boldsymbol{\omega}$ and θ are fundamental properties of fluid motion is made evident in Chapter 4, where it is seen that, taken together, they can be used to determine the velocity field with which they are associated.

1.4 The Big Picture

The motion of fluids is determined by a few main physical principles whose mathematical and physical elucidation forms the central part of this book. At their most fundamental level, the physical ideas that need to be developed concern how the mass, momentum, and internal energy of a material element of fluid varies in time.

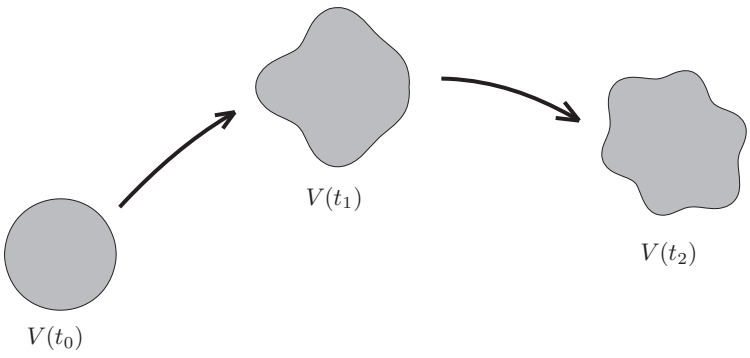


Figure 1.4. Motion of a material fluid element in time with $t_0 < t_1 < t_2$.

A material element, as illustrated in Fig. 1.4, is an identified volume of fluid that retains its identity as it moves through the surrounding fluid even while distorting in various ways. In essence, a material fluid element is the equivalent for fluids of the deformable bodies whose dynamics are studied in the discipline of solid mechanics. The very different constitutive properties of fluids and solids, including the way in which they respond to strain, means that their resulting equations of motion are different, even if the governing physical laws are the same.

A material fluid element can be described mathematically by the volume in space, $V(t)$, that it occupies at time $t \geq t_0$ as it travels downstream from a given initial location coinciding with volume, $V(t_0)$. A particular material element is distinguished from all other elements by the volume it occupies at the initial instant $V(t_0)$. From the field quantities $\mathbf{u}(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$, and $E(\mathbf{x}, t)$, one can specify the mass of the material element at time t as

$$M(t) = \int_{V(t)} \rho(\mathbf{x}, t)\, d\mathbf{x}, \tag{1.6}$$

the momentum as

$$\mathbf{P}(t) = \int_{V(t)} \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)\, d\mathbf{x}, \tag{1.7}$$

and the internal energy as

$$\mathcal{E}(t) = \int_{V(t)} \rho(\mathbf{x}, t)E(\mathbf{x}, t)\, d\mathbf{x}. \tag{1.8}$$

The notation $d\mathbf{x} \equiv dx\, dy\, dz$ for an element of volume in the integration emphasizes that $\mathbf{x}$ is the integration variable. Sometimes, as appropriate in other contexts, the simpler notation dV is used to indicate a volume element. To a large extent, the dynamics of fluid flow is determined by the fundamental laws governing the quantities in Eqs. (1.6)–(1.8).

The first physical law governing the motion of fluids is that of mass conservation, expressed as

$$\frac{dM}{dt} = 0, \tag{1.9}$$

specifying that the mass of a material element does not change in time. Nuclear decay is not being considered here. The second essential physical requirement is that the momentum of the material element should satisfy Newton's second law of

motion, which posits that, in an inertial frame of reference,

$$\frac{d\mathbf{P}}{dt} = \sum_i \mathbf{F}_i, \tag{1.10}$$

where the right-hand side denotes the sum of forces that act on the material volume as it moves. Among the forces are body forces, such as gravity, that effect the entire material element and surface forces acting on the boundary of the material element that arise from its interaction with the surrounding fluid or solid boundaries.

The third essential relation is the first law of thermodynamics, which governs changes in the internal energy of the fluid element as it moves. This may be expressed as

$$\frac{d\mathcal{E}}{dt} = \dot{Q} + \dot{W}, \tag{1.11}$$

where the terms on the right-hand side, respectively, denote the rate at which heat (i.e., internal energy) is added or lost to the material element and the rate at which work is done on the element that directly affects internal energy.

A considerable part of this book is devoted to converting the essential ideas contained in Eqs. (1.9)–(1.11) into the form of differential equations that may be used practically in a variety of ways to solve problems involving fluid flow. To accomplish this goal requires developing a number of supporting physical ideas together with the mathematics that allows the development to proceed. In support of this main objective, it is helpful first to spend a number of chapters introducing ancillary ideas related to such aspects of fluid motion as its kinematics and statics. After the fundamental laws are derived, our interest shifts to examining in detail the solutions to a number of representative flows.

1.5 Vector and Tensor Analysis

A goal of this book is to present many of the equations of fluid dynamics and their derivations in a relatively unencumbered mathematical form. This effort is helped greatly by often relying on direct notation for vectors and tensors, that is, notation lacking the use of indices.[1] We already encountered the difference between direct and index notation in Eq. (1.1), where the first expression for the divergence of the velocity in direct notation is equated to the same quantity in index notation. In fact, the latter is the form the divergence takes specifically for rectangular Cartesian coordinates. In other coordinate systems, such as cylindrical or spherical, the expression would look different, as may be seen in Appendix A, where a number of vector operators as well as the equations of motion are written out in alternative coordinate systems. This point makes clear the value of direct notation, as the vector expression is not specific to any one coordinate system.

A number of other mathematical quantities besides $\nabla \cdot \mathbf{u}$ are encountered in this book, for which both direct and index notation make sense, depending on the context. In fact, it will be seen that sometimes proofs of identities can be accomplished more simply in index notation than in direct notation, and we do not hesitate to take advantage of this when it is appropriate.

[1] An exhaustive and lucid treatment of vector and tensor analysis through direct notation may be found in Gurtin (1981) and Gurtin, Fried & Anand (2010).

This introductory section gives a brief description of some important mathematical entities that play a major role in much of the subsequent development. Additional concepts are introduced as needed in the course of the book, including in Appendix B, where we describe some specialized concepts that are best left out of the main line of development.

1.5.1 Vectors

We have already encountered the velocity field $\mathbf{u}(\mathbf{x}, t)$ that, for any fixed time, is an example of a vector field, that is, a vector function of vector position in three dimensions. At any point $\mathbf{x}$, three orthogonal directions can be specified and a right-handed set of orthogonal unit basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ can be chosen. Such vectors are referred to as being orthonormal, because besides being mutually perpendicular to each other, they are unit vectors. In some contexts, the basis vectors will be indicated as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Orthonormality means that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where

$$\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.12}$$

is the Kronecker delta. Right-handedness means that $\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k$ with (i, j, k) in cyclic order. For a particular choice of basis vectors, any vector field $\mathbf{v}(\mathbf{x})$ has components defined via

$$v_i(\mathbf{x}) \equiv \mathbf{v}(\mathbf{x}) \cdot \mathbf{e}_i, \tag{1.13}$$

and one may write

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3. \tag{1.14}$$

The basis vectors may either be independent of $\mathbf{x}$ position, as in rectangular Cartesian coordinates, or position dependent, as in cylindrical or spherical coordinate systems. For convenience, vectors are sometimes referred to via their components, as in $\mathbf{v} = (v_1, v_2, v_3)$.

The dot or inner product of vectors $\mathbf{v}$ and $\mathbf{w}$ is given in terms of their components by the scalar

$$\mathbf{v} \cdot \mathbf{w} = v_i w_i, \tag{1.15}$$

which represents a projection of either vector in the direction of the other. The magnitude or length of a vector is defined using its components as

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_i v_i}. \tag{1.16}$$

The cross-product of two vectors is the vector with components

$$(\mathbf{v} \times \mathbf{w})_i = \epsilon_{ijk} v_j w_k, \tag{1.17}$$

which may be shown to have orientation normal to the plane formed by $\mathbf{v}$ and $\mathbf{w}$ and magnitude equal to the area of the parallelogram formed by the vectors. To evaluate expressions such as $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ for three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (see Problem 1.6), the identity

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \tag{1.18}$$

is of considerable help.

1.5.2 Tensors

The concept of a vector field is a special case of a more general entity known as the tensor that, in a number of guises, plays a major role in expressing the physics of fluid flow. For example, the stress tensor $\sigma(\mathbf{x}, t)$, which is an example of a second-order tensor, appears in a natural way when accounting for the surface force fields appearing in Newton's law of motion. Though we have not referred to them as tensors, in fact, scalars may be regarded as zeroth-order tensors and vectors as first-order tensors. Although tensors of third, fourth, and higher order exist, here, for simplicity, second-order tensors are referred to as tensors because most of our use of tensors will be of this variety. In this case, our formal definition of a tensor is that it is a linear map from vectors to vectors. In other words, if T is a tensor and $\mathbf{v}$ is a vector, then the quantity $\mathbf{w} = T\mathbf{v}$ is a vector. Linearity in the mapping means that for any two vectors $\mathbf{v}$ and $\mathbf{w}$ and constant α, it follows that

$$T(\alpha\mathbf{v} + \mathbf{w}) = \alpha T\mathbf{v} + T\mathbf{w}. \tag{1.19}$$

This relation means that the vector on the left-hand side resulting from T operating on the vector $\alpha\mathbf{v} + \mathbf{w}$ is equal to the linear combination of vectors on the right-hand side.

Tensors have components, as do vectors. In this case, a tensor T has components T_{ij} defined via the relation

$$T_{ij} = (T\mathbf{e}_j) \cdot \mathbf{e}_i, \tag{1.20}$$

where it is seen that the term in parentheses, namely, $T\mathbf{e}_j$, is a vector, so its dot product with $\mathbf{e}_i$ makes sense, in fact, yielding the scalar component of the tensor T_{ij} on the left-hand side. Whereas a vector has three components, a second-order tensor has nine. It is often convenient to represent vectors as 3×1 matrices and tensors as 3×3 matrices. Like matrices, tensors have a transpose T^t that is defined according to $T^t_{ij} = T_{ji}$.

A useful operation on tensors, which linearly maps them into the real numbers, is the trace, denoted *tr* and defined by

$$trT = T_{ii}, \tag{1.21}$$

that is, the sum of the diagonal components. Analogous to vectors, tensors T and S have an inner product defined as

$$T : S = \mathrm{tr}(T^t S) = T_{ij} S_{ij}, \tag{1.22}$$

where $T^t S$ is a tensor formed from the product of the tensors T^t and S (see Problem 1.8). The identity tensor, denoted as I, has the property that $I\mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$. It is readily shown that the matrix of components of I is the Kronecker delta δ_{ij} and, moreover, that $\mathrm{tr}(I) = 3$.

Tensors may be formed out of vectors via the tensor product. In this, for any two vectors $\mathbf{a}$, $\mathbf{b}$, the quantity $\mathbf{a} \otimes \mathbf{b}$ is a tensor known as the *tensor product* of $\mathbf{a}$ and $\mathbf{b}$. It is defined by the property that for any vector $\mathbf{v}$, it returns a vector aligned in the $\mathbf{a}$ direction by the formula

$$\mathbf{w} = (\mathbf{a} \otimes \mathbf{b})\mathbf{v} \equiv \mathbf{a}(\mathbf{b} \cdot \mathbf{v}). \tag{1.23}$$

Using this definition, it is easy to verify that the components of the tensor $\mathbf{a} \otimes \mathbf{b}$ are given by

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = ((\mathbf{a} \otimes \mathbf{b})\mathbf{e}_j) \cdot \mathbf{e}_i = a_i b_j . \tag{1.24}$$

In fact, it is not hard to show that any tensor T with components T_{ij} can be written as

$$T = T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \tag{1.25}$$

with repeated indices summed.

1.5.3 Skew Tensors

A tensor T that satisfies

$$T^t = T \tag{1.26}$$

is said to be symmetric. Alternatively, a tensor T satisfying

$$T^t = -T \tag{1.27}$$

is said to be skew or antisymmetric. In this case, its components satisfy $T_{ij} = -T_{ji}$ for all values of $1 \leq i, j \leq 3$. Consequently,

$$T_{11} = T_{22} = T_{33} = 0, \tag{1.28}$$

and

$$T_{12} = -T_{21}, \tag{1.29a}$$

$$T_{23} = -T_{32}, \tag{1.29b}$$

$$T_{31} = -T_{13}, \tag{1.29c}$$

so that, in fact, there are only three independent elements in a skew tensor.

Making the definitions

$$\Omega_1 \equiv T_{32} = -T_{23}, \tag{1.30a}$$

$$\Omega_2 \equiv T_{13} = -T_{31}, \tag{1.30b}$$

$$\Omega_3 \equiv T_{21} = -T_{12}, \tag{1.30c}$$

and substituting for the components of T using Eqs. (1.28) and (1.30), it is seen that

$$T = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}. \tag{1.31}$$

For an arbitrary vector $\mathbf{v} = (v_1, v_2, v_3)^t$, a calculation gives

$$T\mathbf{v} = \begin{pmatrix} -\Omega_3 v_2 + \Omega_2 v_3 \\ \Omega_3 v_1 - \Omega_1 v_3 \\ -\Omega_2 v_1 + \Omega_1 v_2 \end{pmatrix}. \tag{1.32}$$