Granularity Theory with Applications to Finance and Insurance

The recent financial crisis has heightened the need for appropriate methodologies for managing and monitoring complex risks in financial markets. The measurement, management, and regulation of risks in portfolios composed of credits, credit derivatives, or life insurance contracts is difficult because of the nonlinearities of risk models, dependencies between individual risks, and the number of contracts in large portfolios. Granularity principle was introduced in the Basel regulations for credit risk to solve these difficulties in computing capital reserves. In this book, authors Patrick Gagliardini and Christian Gouriéroux provide the first comprehensive overview of the granularity theory and illustrate its usefulness for a variety of problems related to risk analysis, statistical estimation, and derivative pricing in finance and insurance. They show how the granularity principle leads to analytical formulas for risk analysis that are simple to implement and accurate even when the portfolio size is large.

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GRANULARITY THEORY WITH APPLICATIONS TO FINANCE AND INSURANCE

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Preface

This book provides the first comprehensive overview of granularity theory and illustrates its potential for risk analysis in finance and insurance.

The Granularity Principle

The recent financial crisis has heightened the need for appropriate methodologies to control and regulate risks in financial markets. The balance sheets of banks and insurance companies contain large portfolios of individual risks that correspond to financial securities, such as stocks and corporate or sovereign bonds, as well as individual contracts, such as corporate loans, household mortgages, and life insurance contracts. Risk analysis in such large portfolios is made difficult by the nonlinearities of the risk models, the dependencies between the individual risks, and the large sizes of the portfolios, which can include several thousand assets and contracts. The nonlinearities are induced, for instance, by the qualitative nature of the risks associated with default, rating migration, and prepayment for credit portfolios, or with mortality and lapse for life insurance portfolios. The dependencies between the individual securities and contracts are caused by systematic risk factors that affect the random payoffs of the individual assets. Systematic risks cannot be diversified even when the size of the portfolio becomes infinitely large. The consequence of these difficulties is that standard portfolio risk measures, such as the Value-at-Risk (VaR), cannot be computed analytically for realistic risk models. The portfolio VaR corresponds to the quantile of the portfolio loss distribution at a given percentile level; that is, the loss that is exceeded only with a given small probability. The VaR is currently the basis for the computation of the required capital, and hence of the reserves. Similar difficulties in analytical tractability are encountered when estimating the unknown parameters in risk models and when pricing derivative assets written on large portfolios of individual risks, such as basket default swaps. A basket default swap (BDS) pays off \$1 at maturity, if the percentage of defaults in a large portfolio of loans is larger than a given threshold.

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Efficient simulation-based techniques for computing risk measures, estimating model parameters, and pricing basket derivatives have been developed in the literature. However, these techniques can be very time consuming and can be insufficient, for instance, when fast intraday computations of the reserves are required or when investigating the effects on portfolio risk from stress scenarios of model parameters and risk factors. For these tasks, analytical (i.e., closed-form) approximations of the risk quantities of interest provide a valuable input. In this book, we focus on the class of analytical approximations that are derived from the so-called granularity principle.

The granularity principle was introduced in the Basel 2 regulation for credit risk for the purpose of facilitating the internal computation of the reserves by financial institutions. The granularity principle proceeds according to three steps.

- 1. First, the modeling step considers a risk factor model (RFM) that relates the payoffs/losses of the individual assets in the portfolio to systematic risk factors and unsystematic (i.e., idiosyncratic) risks. Systematic risk factors represent the undiversifiable sources of uncertainty that are common to all individual assets in the portfolio and that introduce dependencies between the individual risks.
- 2. Second, the RFM is applied to a virtual portfolio of infinite size, which represents the ideal limit of a very large portfolio. In this asymptotic risk factor model (ARFM), all idiosyncratic risks are perfectly diversified, and the only remaining uncertainty governing the individual risks is through systematic risk factors. In general this simplification allows the derivation of explicit formulas for the portfolio risk measures (standardized per unit of contracts in the portfolio) and thus for the required capital. The value of a risk measure for a portfolio of infinite size is called the cross-sectional asymptotic (CSA) risk measure.
- 3. Third, for the real portfolio with a finite but large size, closed-form approximations of the risk measures are derived by an asymptotic expansion around the ARFM. These approximations are given by the CSA risk measure plus an adjustment term, called the granularity adjustment (GA). The GA is of order 1/n, where *n* denotes the number of individual contracts in the portfolio. The GA accounts for the remaining idiosyncratic risks and for their interaction with the systematic risks in a portfolio of finite size.

Very often in practice the third step is omitted in the computation of the reserves. This omission can lead to a significant underestimation of the required capital. In fact, as we see in the illustrations in the book, the GA can contribute to adjustments of several percentage points in the risk measure even for portfolios with thousands of assets, especially in a dynamic multifactor framework.

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Although granularity theory was originally motivated by the problem of computing risk measures for large portfolios, the same principle can be followed to address a variety of different but related issues. For instance, it can be applied to estimate the unknown parameters in the RFM using a large panel of individual risk histories, for filtering the unobservable values of systematic risk factors, or for pricing basket derivatives written on large portfolios. For these tasks, the basic concept of performing an asymptotic expansion around a large cross-sectional limit is applied to different objects of interests: a likelihood function, a filtering distribution, and a derivative price. Thus, granularity theory provides a tractable framework for risk analysis of large portfolios that integrates the effects of systematic risks, idiosyncratic risks, uncertainty on the model parameters, and unobservability of the states.

The Terminology

The granularity terminology was introduced by Gordy¹ by analogy with the terminology used in physics or in photography. A system is infinitely fine grained, or granular, if it can be broken down into small parts, or grains, of similar size, such that no grain has a significant effect on the entire system. In the application to credit, the system is a loan portfolio, and the grains are the individual loans. Then, the portfolio is infinitely granular if the loans have similar exposures and there exists no loan carrying a systemic risk. In a probabilistic setting, the individual risks associated with the grains are not necessarily independent. More precisely, in an infinitely fine-grained system the dependencies and heterogeneities across the grains are such that, conditionally on some systematic risk factor, the grains are independent and identically distributed (i.i.d.). This conditional i.i.d. property is more general than what might appear initially because the factor can be dynamic and multidimensional (even infinite-dimensional), and it is compatible with random individual effects. The conditional i.i.d. property allows application of the standard stochastic limit theorems, such as the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT), conditionally on the realization of the systematic risk factors. This property is the key to deriving the asymptotic expansions underlying the granularity principle.

Topics Covered by This Book

This book covers the computation of risk measures, the estimation of model parameters, the prediction of unobservable factors, and the pricing of basket derivatives for a large population of individual risks by means of granularity

¹ Gordy, M. (2003). "A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules," *Journal of Financial Intermediation*, 12, 199–232.

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theory. In each chapter, selected numerical and empirical applications in finance and insurance illustrate the theoretical results. The organization of the material reflects our attempt to present the results in a unifying framework that highlights the methodological similarities among apparently different problems. We also favor a pedagogical exposition – building the theory from the simpler to the more sophisticated setting. These principles explain why the sequence of the topics treated in the various chapters does not reflect the historical development of granularity theory, for instance, the chapter on risk measures is the last one.

Chapter 1 introduces the general modeling framework used throughout this book, namely that of a large homogeneous panel of histories of individual risks. These individual risks correspond to the assets and contracts in a portfolio. We explain the standard stochastic limit theorems - the LLN and the CLT and why they cannot be applied directly when the individual histories feature dependencies caused by systematic risk factors. However, the infinitely granular nature of a homogeneous population allows us to apply the limit theorems conditionally on the realization of the systematic risk factors. Chapters 2 and 3 consider static RFM (i.e., the systematic and unsystematic risks are i.i.d. across time). Specifically, Chapter 2 introduces the linear static RFM and presents the granularity adjustments for parameter estimation and portfolio management. Chapter 3 considers nonlinear static RFM for qualitative individual risks. An important specification in this class is the single risk factor (SRF) model for default that is based on a multifirm version of the Merton structural model. The individual risk variables correspond to default indicators of the firms, and a systematic risk factor drives the probability of default (PD) of any single firm. We present several empirical illustrations, including the estimation of a factor model for corporate default in the United States, and of a stochastic intensity model for longevity risk on French mortality data.

In Chapters 4-6 we move to dynamic (nonlinear) RFM. We distinguish between the microdynamic - the dependence of the individual risks on their own lagged values conditionally on the factor path – and the macrodynamic, namely the serial persistence of the common factor. Specifically, Chapter 4 focuses on the estimation of the RFM parameters based on a large panel with T time observations for n individuals. Maximum likelihood (ML) estimation in such a nonlinear state space model is complicated because the likelihood function involves a large-dimensional integral with respect to the factor path. We show how the granularity theory leads to asymptotic closed-form approximations of the likelihood function for large cross-sectional dimension n. By maximizing the approximate likelihood function we get easy-to-compute estimators that are asymptotically efficient in a large n and T asymptotics. We illustrate the methodology with an application to estimation of a stochastic migration model for corporate rating using S&P data. In this model, rating migration correlation across firms is introduced by systematic unobservable factors that drive the stochastic transition matrices.

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Chapter 5 concerns the prediction and filtering of the unobservable value of the systematic risk factors given the available individual risk histories. The predictive and filtering distributions of the unobservable common factor in a nonlinear state space model are analytically intractable. However, in our framework granularity theory can be used to derive closed-form Gaussian approximations of these distributions for large cross-sectional size. Although the value of the common factor becomes ex-post observable from well-chosen cross-sectional aggregates in the limit of an infinite portfolio size, the granularity adjustment describes the bias and uncertainty on the factor value for finite but large n. This result finds applications for basket derivative pricing. Indeed, the payoff of a basket derivative such as a BDS is written on the portfolio default frequency, which is a proxy of (a transformation of) the systematic risk factor, namely the default probability. We show in Chapter 5 that the pricing of basket derivatives such as BDS can be reduced to the pricing of (fictitious) derivatives written on the value of the systematic risk factor. The approximate pricing of the latter derivatives can be performed under the Gaussian predictive and filtering distributions of the factor.

Chapter 6 focuses on the granularity adjustment for risk measures. We first show that the GA for the VaR is the input to derive the GA for a broad class of risk measures, known as distortion risk measures (DRM), that include, for instance, the expected shortfall (ES). The ES is a coherent risk measure that provides the average loss amount when a loss above the level of VaR occurs. We then derive the GA for the VaR in both the static and dynamic RFM with single or multiple risk factors. In a dynamic RFM, the VaR is a quantile of the conditional portfolio loss distribution given past individual risk histories. The GA consists of two components. The first GA component is an adjustment for the idiosyncratic risk (which is not completely diversified in a portfolio with a large but finite size) and for its interaction with systematic risk. The second GA component accounts for the filtering of the unobservable systematic risk factor from the individual risk histories. We illustrate the patterns and dynamics of the GA with an application to a dynamic model with stochastic default and recovery. This example shows that the GA VaR is larger and features a smoother pattern across time than the CSA VaR. Thus, accounting for GA leads to a larger and more stable level of reserves.

The book is intended for graduate students, researchers, and professionals working in the areas of risk control and regulation. We tried to reach a balance between emphasis on the financial applications motivating this book and on the theoretical tools in probability and econometrics necessary for a sufficiently rigorous presentation of the results. A minimal background in statistics and finance is required at the level of introductory master courses in these subjects. However, two review chapters help the reader reach this level. These chapters cover basic material in econometrics (such as linear panel models, principal component analysis [PCA], and the Kalman filter) and finance theory (such as portfolio management, arbitrage theory, and risk measures).

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Instructors who use this book might find it difficult or unnecessary to cover all included material. Selecting certain chapters is possible depending on one's specific purposes. For instance, an introduction to granularity theory at a basic level can be limited to static RFM (Chapters 1–3 and Sections 6.1–6.3). A short course focusing on the granularity adjustment for risk measures can be based on Chapters 1 and 6 and Section 5.1.

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List of Acronyms

AAO	absence of arbitrage opportunities
APT	arbitrage pricing theory
ARFM	asymptotic risk factor model
BCBS	Basel Committee on Banking Supervision
BDS	basket default swap
c.d.f.	cumulative distribution function
CDO	collateralized debt obligation
CDS	credit default swap
CLT	Central Limit Theorem
CSA	cross-sectional asymptotic
DRM	distortion risk measure
ELGD	expected loss given default
ES	expected shortfall
GA	granularity adjustment
i.i.d.	independent identically distributed
L&P	loss and profit
LGD	loss given default
LLN	Law of Large Numbers
LSRF	linear single risk factor
ML	maximum likelihood
MLS	mortality linked securities
OLS	ordinary least squares
P&L	profit and loss
PaR	population at risk
PCA	principal component analysis
PD	probability of default
p.d.f.	probability distribution function
RFM	risk factor model
SDF	stochastic discount factor

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- SRF single risk factor
- SVD singular value decomposition
- VaR value at risk
- VAR vector autoregression
- VGA variance granularity adjusted