Part I

Instantons

1

Instantons in Quantum Mechanics

1.1 Introduction

In this chapter we start our study of non-perturbative effects by looking at the simplest case: Quantum Mechanics (QM), which can be regarded as a QFT in one dimension. We will focus on effects due to *instantons*, i.e. to non-trivial solutions to the Euclidean equations of motion (EOM). Typically, if g is the coupling constant, these effects go like

$$e^{-A/g}. \tag{1.1.1}$$

Notice that this effect is still small if g is small. However, it is completely invisible in perturbation theory, since it displays an essential singularity at $g = 0$.

Instanton effects are responsible for one of the most important quantum mechanical effects: tunneling through a potential barrier. This effect changes qualitatively the structure of the quantum vacuum. In a potential with a perturbative ground state degeneracy, like the one shown on the left hand side of Fig. 1.1, tunneling effects lift the degeneracy: there is a single ground state, and the energy difference between the ground state and the first excited state is of the form (1.1.1),

$$E_1(g) - E_0(g) \sim e^{-A/g}. \tag{1.1.2}$$

In a potential with an unstable or "false" vacuum, like the one shown on the right hand side of Fig. 1.1, states trapped in the false vacuum will eventually decay due to tunneling effects. This means in particular that the ground state energy associated to this vacuum has a small imaginary part,

$$E_0(g) = \mathrm{Re}\, E_0(g) + \mathrm{i}\, \mathrm{Im}\, E_0(g), \qquad \mathrm{Im}\, E_0(g) \sim e^{-A/g}, \tag{1.1.3}$$

which also has the dependence on g typical of an instanton effect and is invisible in perturbation theory.

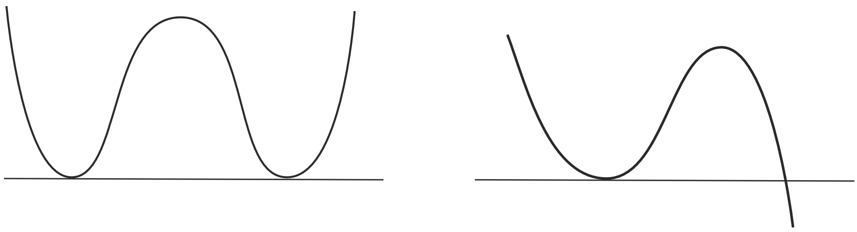

Figure 1.1 Two quantum mechanical potentials where instanton effects change qualitatively our understanding of the vacuum structure.

In this chapter, in order to understand this type of non-perturbative effect in detail, we will focus on observables which vanish in conventional perturbation theory, but have contributions due to instantons. In the case of vacuum decay, this will be the inverse lifetime of the particle; in the case of degenerate vacua, it will be the energy splitting (1.1.2). In our discussion we will focus on one-dimensional problems, where one can find very explicit results for instanton effects.

1.2 Quantum Mechanics as a one-dimensional field theory

In this chapter we will consider quantum systems in one dimension, with a Hamiltonian of the form,

$$H = \frac{1}{2}p^2 + W(q), \tag{1.2.1}$$

where $W(q)$ is the potential. We will set $\hbar = 1$. If this Hamiltonian supports bound states, one basic question to ask is what is the energy of the ground state. This can of course be addressed by elementary methods, like stationary perturbation theory, but we want to formulate the problem in the language of path integrals, so that the intuition gained in this way can be applied to QFTs. The ground state energy of the quantum mechanical system described by (1.2.1) can be extracted from the small temperature behavior of the thermal partition function,

$$Z(\beta) = \mathrm{tr}\, \mathrm{e}^{-\beta H}. \tag{1.2.2}$$

Indeed, if we have a non-degenerate, discrete spectrum with energies

$$E_0 < E_1 < E_2 < \cdots , \tag{1.2.3}$$

the thermal partition function can be written as

$$Z(\beta) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n}, \tag{1.2.4}$$

therefore

$$E_0 = -\lim_{\beta \to \infty} \frac{1}{\beta} \log Z(\beta). \tag{1.2.5}$$

On the other hand, the thermal partition function admits a path integral representation in terms of the Euclidean theory, in which we perform a Wick rotation to imaginary time

$$t \to -\mathrm{i}t, \tag{1.2.6}$$

and, because of the trace in (1.2.2), we have to consider *periodic* trajectories $q(t)$ in imaginary time,

$$q(-\beta/2) = q(\beta/2), \tag{1.2.7}$$

where β is the period of the motion. After Wick rotation, the path integral involves the Euclidean action $S(q)$,

$$S(q) = \int_{-\beta/2}^{\beta/2} \mathrm{d}t \left[\frac{1}{2}(\dot{q}(t))^2 + W(q(t)) \right]. \tag{1.2.8}$$

The thermal path integral is then given by

$$Z(\beta) = \int \mathcal{D}[q(t)] \mathrm{e}^{-S(q)}, \tag{1.2.9}$$

where the integration is performed over periodic trajectories.

We note that the Euclidean action can be regarded as an action in Lagrangian mechanics,

$$S(q) = \int_{-\beta/2}^{\beta/2} \mathrm{d}t \left[\frac{1}{2}(\dot{q}(t))^2 - V(q) \right], \tag{1.2.10}$$

where the potential is

$$V(q) = -W(q), \tag{1.2.11}$$

i.e. it is the inverted potential of the original problem.

It is possible to compute the ground state energy by using Feynman diagrams. We will assume that the potential $W(q)$ is of the form

$$W(q) = \frac{1}{2} q^2 + W_{\rm int}(q), \tag{1.2.12}$$

where $W_{\rm int}(q)$ is an interaction term. Then, the path integral defining Z can be computed in perturbation theory by expanding in $W_{\rm int}(q)$. For concreteness, let us assume that we have a quartic interaction (i.e. an anharmonic, quartic oscillator)

$$W_{\rm int}(q) = \frac{g}{4} q^4. \tag{1.2.13}$$

At leading order in g, we find,

$$Z(\beta) = Z_{\mathrm{G}}(\beta)\left(1 - \frac{g}{4}\int \mathrm{d}\tau \, \langle q(\tau)q(\tau)q(\tau)q(\tau)\rangle_{\mathrm{G}} + \cdots\right). \tag{1.2.14}$$

Here, $Z_{\mathrm{G}}(\beta)$ is the Euclidean partition function of the theory with the unperturbed Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2, \tag{1.2.15}$$

which is nothing but the thermal partition function of a harmonic oscillator with normalized frequency $\omega = 1$,

$$Z_{\mathrm{G}}(\beta) = \frac{1}{2\sinh\left(\frac{\beta}{2}\right)}. \tag{1.2.16}$$

The subscript G indicates that, from the point of view of the path integral, this is a Gaussian theory. The bracket $\langle \cdots \rangle_{\mathrm{G}}$ denotes a normalized vacuum expectation value (vev) in this Gaussian theory, which can be computed by using Wick's theorem. As usual, the calculation can be organized in terms of Feynman diagrams. We will actually work in the limit in which $\beta \to \infty$, since in this limit many features are simpler, for example the form of the propagator, which reads

$$\langle q(\tau)q(\tau')\rangle_{\mathrm{G}} = \int \frac{\mathrm{d}p}{2\pi}\frac{\mathrm{e}^{\mathrm{i}p(\tau-\tau')}}{p^2+1} = \frac{\mathrm{e}^{-|\tau-\tau'|}}{2}. \tag{1.2.17}$$

The Feynman rules are illustrated in Fig. 1.2. Since we want to calculate $\log Z(\beta)$, only *connected vacuum diagrams* contribute. In the limit $\beta \to \infty$, the quantity $\log Z(\beta)$ should be given by an overall factor of β, times a β-independent constant, as follows from (1.2.5). Diagrammatically, this is due to the following: the standard

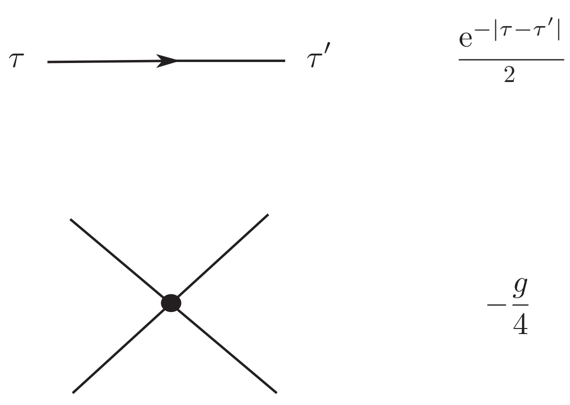

Figure 1.2 Feynman rules for the quantum mechanical quartic oscillator.

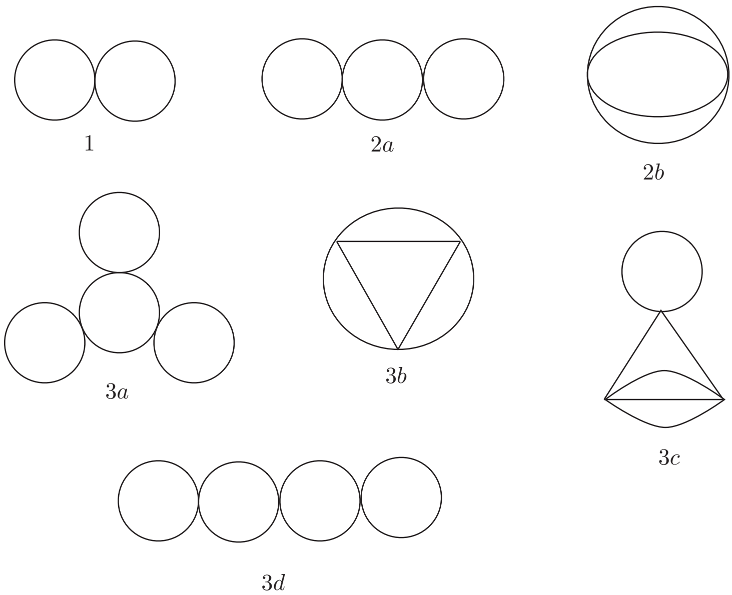

Figure 1.3 Feynman diagrams contributing to the ground state energy of the quartic oscillator up to order g^3.

Feynman rules in position space lead to k integrations, where k is the number of vertices in the diagram. One of these integrations just gives as an overall factor the "volume" of spacetime, and this is the overall factor of β. Therefore, in order to extract $E_0(g)$, we can just perform $k-1$ integrations over $\mathbb{R}$.

It follows that the ground state energy has the following perturbative expansion:

$$E_0(g) = \frac{1}{2} + \sum_{k=1}^{\infty} a_k \left(\frac{g}{4}\right)^k, \tag{1.2.18}$$

where a_k can be computed diagrammatically as follows. Let $\mathcal{A}_k^{(c)}$ be the set of independent, connected quartic diagrams with k vertices. For $k = 1, 2, 3$, these diagrams are shown in Fig. 1.3. Then,

$$a_k = \sum_{\Gamma \in \mathcal{A}_k^{(c)}} s_\Gamma \mathcal{I}_\Gamma, \tag{1.2.19}$$

where s_Γ is the multiplicity of the graph Γ and $\mathcal{I}_\Gamma$ is the corresponding Feynman integral. Here, the multiplicity is simply the number of contractions which lead to the same topological graph Γ, and we can interpret it as the "number" of graphs

Table 1.1 *Multiplicities of the Feynman diagrams in Fig. 1.3*

Diagram	1	2*a*	2*b*	3*a*	3*b*	3*c*	3*d*
Multiplicity	3	36	12	288	288	576	432

with the topological structure Γ (in the literature one sometimes finds other definitions of the multiplicity, differing typically in the normalization of the coupling constant).

It is now straightforward to calculate $E_0(g)$ to order g^3. The multiplicities of the diagrams shown in Fig. 1.3 are given in Table 1.1. These numbers can be checked by taking into account that the total symmetry factor for a connected diagram with k quartic vertices is given by

$$\frac{1}{k!}\langle (x^4)^k\rangle^{(c)}, \tag{1.2.20}$$

where

$$\langle (x^4)^k\rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2} x^{4k}}{\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2}}. \tag{1.2.21}$$

is the Gaussian average. By Wick's theorem, this counts all possible pairings among k four-vertices, and we have to take the connected piece. Using that

$$\langle x^{2k}\rangle = (2k-1)!! = \frac{(2k)!}{2^k k!} \tag{1.2.22}$$

we find, for example,

$$\begin{aligned} \langle x^4\rangle^{(c)} &= \langle x^4\rangle = 3,\\ \frac{1}{2!}\langle (x^4)^2\rangle^{(c)} &= \frac{1}{2}\left(\langle (x^4)^2\rangle - \langle x^4\rangle^2\right) = 48, \end{aligned} \tag{1.2.23}$$

in agreement with the results shown in Table 1.1. Putting together the Feynman integrals with the multiplicities, we find, for the different diagrams of Fig. 1.3,

$$\begin{aligned} 1 &: \quad \frac{3}{4}\\ 2a &: \quad -\frac{36}{16}\int_{-\infty}^{\infty} \mathrm{e}^{-2|\tau|}\mathrm{d}\tau = -\frac{36}{16},\\ 2b &: \quad -\frac{12}{16}\int_{-\infty}^{\infty} \mathrm{e}^{-4|\tau|}\mathrm{d}\tau = -\frac{12}{16}\cdot\frac{1}{2},\\ 3a &: \quad \frac{288}{64}\int_{-\infty}^{\infty} \mathrm{e}^{-|\tau_1|-|\tau_2|-|\tau_1-\tau_2|}\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \frac{288}{64}\cdot\frac{3}{2}, \end{aligned} \tag{1.2.24}$$

$$
\begin{aligned}
3b: &\quad \frac{288}{64}\int_{-\infty}^{\infty} \mathrm{e}^{-2|\tau_1|-2|\tau_2|-2|\tau_1-\tau_2|}\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \frac{288}{64}\cdot\frac{3}{8},\\
3c: &\quad \frac{576}{64}\int_{-\infty}^{\infty} \mathrm{e}^{-|\tau_1-\tau_2|-|\tau_1|-3|\tau_2|}\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \frac{576}{64}\cdot\frac{5}{8},\\
3d: &\quad \frac{432}{64}\int_{-\infty}^{\infty} \mathrm{e}^{-2|\tau_1-\tau_2|-2|\tau_2|}\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \frac{432}{64}.
\end{aligned}
$$

This gives,

$$
E_0(g) = \frac{1}{2} + \frac{3}{4}\left(\frac{g}{4}\right) - \frac{21}{8}\left(\frac{g}{4}\right)^2 + \frac{333}{16}\left(\frac{g}{4}\right)^3 + \mathcal{O}(g^4). \tag{1.2.25}
$$

In Chapter 4 we will be interested in understanding this series in detail, and in particular we will look at the behavior of its coefficients at high order. The method of Feynman diagrams, although it emphasizes the parallelism with field theory, is not the most efficient method to use in order to generate the perturbative series for the ground state. In order to do that, it is better to use the Schrödinger equation

$$
\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{2} + \frac{gx^4}{4}\right)\psi(x) = E_0(g)\psi(x). \tag{1.2.26}
$$

We know that, for $g = 0$, the solution to this equation is the ground state of the harmonic oscillator, which is just the Gaussian $\mathrm{e}^{-x^2/2}$. We will then write down an ansatz for the solution of the form

$$
\psi(x) = \mathrm{e}^{-x^2/2}\sum_{n=0}^{\infty}\left(\frac{g}{4}\right)^n B_n(x), \qquad B_0(x) = 1. \tag{1.2.27}
$$

Plugging this ansatz into the above equation, and writing the energy as in (1.2.18), we find the following recursive equation for the $B_k(x)$ and the a_k:

$$
xB_k'(x) - \frac{1}{2}B_k''(x) + x^4 B_{k-1}(x) = \sum_{p=0}^{k} a_{k-p}B_p(x). \tag{1.2.28}
$$

To solve this recursion, we further write

$$
B_i(x) = \sum_{j=1}^{2i} x^{2j}(-1)^i B_{i,j}. \tag{1.2.29}
$$

By looking at the term of degree zero in (1.2.28), we find that

$$
a_k = (-1)^{k+1}B_{k,1}. \tag{1.2.30}
$$

The coefficients $B_{i,j}$ satisfy the recursion relation

$$
2jB_{i,j} = (j+1)(2j+1)B_{i,j+1} + B_{i-1,j-2} - \sum_{p=1}^{i-1} B_{i-p,1}B_{p,j}. \tag{1.2.31}
$$

This recursion can be easily solved to high orders, and one finds for the very first coefficients,

$$a_1 = \frac{3}{4}, \qquad a_2 = -\frac{21}{8}, \qquad a_3 = \frac{333}{16}, \qquad a_4 = -\frac{30885}{128}, \tag{1.2.32}$$

in agreement with the Feynman diagram calculation (1.2.25).

1.3 Unstable vacua in Quantum Mechanics

Most quantities of interest in a quantum theory will have both perturbative and non-perturbative contributions. For small coupling, perturbative contributions are typically dominant. Therefore, in order to better understand the idiosyncrasies of non-perturbative effects in quantum theory, it is convenient to focus on quantities which vanish in perturbation theory.

A situation where non-perturbative effects dominate the physics is the case of unstable minima in QM. Let us consider a one-dimensional potential $W(q)$ which has a relative minimum at the origin $q = 0$. Near this minimum, the potential is of the form

$$W(q) \approx \frac{1}{2}q^2 + \mathcal{O}(g), \tag{1.3.1}$$

where g is a coupling constant which gives the strength of the anharmonicity. Examples of such a situation are the cubic potential

$$W(q) = \frac{1}{2}q^2 - gq^3, \tag{1.3.2}$$

which is depicted in Fig. 1.4 (left), and the inverted quartic potential

$$W(q) = \frac{q^2}{2} + \frac{g}{4}q^4, \qquad g = -\lambda, \quad \lambda > 0, \tag{1.3.3}$$

which is shown on the right hand side of Fig. 1.4. It is clear that these potentials do not admit bound states, since a particle trapped near the minimum of the potential at $q = 0$ will eventually decay by tunneling through the barrier. However, this is *a priori* not detected by doing conventional stationary perturbation theory in the coupling constant g or λ: in both cases, one finds an infinite power series for the energy of, say, the ground state. In particular, for the inverted quartic oscillator, this series is obtained from (1.2.25) by simply setting $g = -\lambda$:

$$E_0(\lambda) = \frac{1}{2} - \frac{3}{4}\left(\frac{\lambda}{4}\right) - \frac{21}{8}\left(\frac{\lambda}{4}\right)^2 - \cdots . \tag{1.3.4}$$

This is then a situation where perturbation theory is unable to describe the essential physics of the problem. What are the interesting quantities that can be computed