1

Preliminary ideas

1.1 Why is calculus hard?

Mathematicians find mathematics hard and are not surprised or dismayed if it takes them a long time and a lot of hard work to understand a piece of mathematics. On the other hand, most of them would agree that the only reason we find mathematics hard is that we are stupid.

The basic ideas of the calculus, like the basic ideas of the rest of mathematics, are easy (how else would a bunch of apes fresh out of the trees be able to find them?), but calculus requires a lot of work to master (after all, we are just a bunch of apes fresh out of the trees). Here is a list of some of the difficulties facing the reader.

Mathematics is a 'ladder subject'. If you are taught history at school and you pay no attention during the year spent studying Elizabethan England, you will get bad grades for that year, but you will not be at a disadvantage next year when studying the American Civil War. In mathematics, each topic depends on the previous topic and you cannot miss out too much.

The ladder described in this book has many rungs and it will be a very rare reader who starts without any knowledge of the calculus and manages to struggle though to the end. (On the other hand, some readers will be familiar with the topics in the earlier chapters and will, I hope, be able to enjoy the final chapters.) Please do not be discouraged if you cannot understand everything; experience shows that if you struggle hard with a topic, even if unsuccessfully, it will be much easier to deal with when you study it again.[1]

[1] I remember being reduced to tears when studying Cartesian tensors for the first time. I cannot now understand how I could possibly have found them difficult.

Mathematics involves deferred gratification. Humans are happy to do A in order to obtain B. The are less happy to do A in order to to do B, to do B in order to do C and then to do C in order to obtain D. Results in mathematics frequently require several preliminary steps whose purpose may not be immediately apparent. In Chapter 2 we spend a long time discussing the integral and the fundamental theorem of the calculus. It is only at the end of the chapter that we get our first payback in the form of a solution to an interesting problem.

Mathematics needs practice. I would love to write music like Rossini. My university library contains many books on the theory of music and the art of composing, but I know that, however many books I read, I will not be able to write music. A composer needs to know the properties of musical instruments, and to know the properties of instruments you need to play at least one instrument well. To play an instrument well requires years of practice.

Each stage of mathematics requires fluency in the previous stage and this can only be acquired by hours of practice, working through more or less routine examples. Although this book contains some exercises[2] it contains nowhere near enough. If the reader does not expect to get practice elsewhere, the first volume of *An Analytical Calculus* by A. E. Maxwell [5] provides excellent exercises in a rather less off-putting format than the standard 'door stop' text book. (However, any respectable calculus book will do.)

This is a first look and not a complete story. As I hope to make clear, the calculus presented in this book is not a complete theory, but deals with 'well behaved objects' without giving a test for 'good behaviour'. This does not prevent it from being a very powerful tool for the investigation of the physical world, but is unsatisfactory both from a philosophical and a mathematical point of view. In the final chapter, I discuss the way in which the first university course in analysis resolves these problems. I shall refer to the calculus described in the book as 'the old calculus' and to the calculus as studied in university analysis courses as 'the new calculus' or 'analysis'.[3]

D'Alembert is supposed to have encouraged his students with the cry 'Allez en avant et la foi vous viendra' (push on and faith will come to you). My ideal

[2] Sketch solutions to most of the exercises can be found on my home page accessible at http://www.dpmms.cam.ac.uk/∼twk/. I have marked a few exercises with a •. These are less central to the exposition. Some of them are quite long or require some thought.

[3] Whenever the reader sees 'well behaved' she can think of the words 'terms and conditions apply' which appear at the end of advertisements. In the 'old calculus' we know that 'terms and conditions apply', but we do not know exactly what they are. In the 'new analysis' they are spelled out in detail.

reader will be prepared to accept my account *on a provisional basis*, but be prepared to begin again from scratch when she meets rigorous analysis.

It is a very bad idea to disbelieve everything that your teachers tell you and a good idea to accept everything that your teachers tell you. However, it is an even better idea to accept that, though most of what you are taught is correct, it is sometimes over-simplified and may occasionally turn out to be mistaken.

The calculus involves new words and symbols. The ideas of the calculus are not arbitrary, but the names given to the new objects and the symbols used are. At the simplest level, the reader will need to recognise the Greek letter δ (pronounced 'delta') and learn a new meaning for the word 'function'. At a higher level, she will need to accept that the names and notations used reflect choices made by many different mathematicians, with many different views of their subject, speaking many languages at many different times over the past 350 years. If we could start with a clean sheet, we would probably make different choices (just as, if we could redesign the standard keyboard, we would probably change the position of the letters). However, we wish to talk with other mathematicians, so we must adopt their language.

My British accent. The theory and practice of calculus are international. The teaching of calculus varies widely from country to country, often reflecting the views of some long dead charismatic educationalist or successful textbook writer. In some countries, calculus is routinely taught at school level whilst, in others, it is strictly reserved for university. Several countries use calculus as an academic filter, a coarse filter in those countries with a strong egalitarian tradition, a fine one in those with an elitist bent.[4]

I was brought up under a system, very common in twentieth-century Western Europe, where calculus was taught as a computational tool in the last two years of school and rigorous calculus was taught in the first year of university. Some of the discussion in the introduction and the final chapter reflects my background, but I do not think this should trouble the reader.

A shortage of letters. The calculus covers so many topics that we run into a shortage of letters. Mathematicians have dealt with this partly by introducing new alphabets and fonts giving us A, a, α, $\mathbf{A}$, $\mathbf{a}$, $\aleph$, $\mathcal{A}$, A, a, $\mathbb{A}$, $\mathfrak{A}$, $\mathfrak{a}$, ... Rather than learn a slew of new symbols, I think that my readers will prefer to accept

[4] In some cultures, passing an examination in calculus is believed to have the same magical effect that passing an examination in Latin is believed to have in England.

that r will sometimes be an integer[5] and sometimes the radius of a circle, and that the same letter will be used to represent different things in different places.

But it is beautiful. Hill walking is hard work, but the views are splendid and the exercise is invigorating. The calculus is one of the great achievements of mankind and one of the most rewarding to study.

1.2 A simple trick

If you ask a mathematician the value of $20\,019 \times 300\,016$, she will instantly reply that 'to the zeroth order' the answer is $6\,000\,000\,000$. By this, she means that $20\,019$ is very close to $20\,000$ and $300\,016$ is very close to $300\,000$ so $20\,019 \times 300\,016$ is very close to $20\,000 \times 300\,000 = 6\,000\,000\,000$.

Now suppose we ask her to be a little more accurate. She will be only a little slower to reply that 'to the first order' the answer is $6\,006\,020\,000$. How did she obtain her answer? In effect, she observed that

$$\begin{aligned} 20\,019 \times 300\,016 &= (20\,000 + 19) \times (300\,000 + 16) \\ &= 20\,000 \times 300\,000 + 19 \times 300\,000 + 16 \times 20\,000 + 19 \\ &\quad \times 16. \end{aligned}$$

The first term consists of two very large numbers multiplied together, the second and third terms consist of a small number multiplying a large number and the fourth term consists of two small numbers multiplied together. The first term is thus much bigger than any of the other terms and so, looking only at this term, we obtain

$$\begin{aligned} 20\,019 \times 30\,0016 &= (20\,000 + 19) \times (300\,000 + 16) \\ &\approx 20\,000 \times 300\,000 = 6\,000\,000\,000 \end{aligned}$$

'to the zeroth order'.[6] The second and third terms are small compared with the first term, but large compared with the fourth term, so, to obtain a more accurate result, we now look at the first three terms to obtain

$$\begin{aligned} 20\,019 \times 300\,016 &= (20\,000 + 19) \times (300\,000 + 16) \\ &\approx 20\,000 \times 300\,000 + 19 \times 300\,000 + 16 \times 20\,000 = 6\,006\,020\,000 \end{aligned}$$

'to the first order'.

[5] An integer is a positive or negative 'whole number'.
[6] The symbol $\approx$ is read as 'is approximately equal to' and in practice means 'this should be pretty close to, but we are not on oath.'

If we do the full calculation, we obtain

$$20\,019 \times 300\,016 = 6\,006\,020\,304,$$

so the 'zeroth order approximation' was pretty good and the 'first order approximation' was excellent.

It is very hard to find new ideas and so, once a mathematician has found a new idea, she tries to push it as far as it will go. Let us look at the structure of our argument. We are interested in multiplying $(a + \delta a)$ by $(b + \delta b)$, where δa and δb are very small compared to a and b.

Notational remark. By long tradition, δa (pronounced 'delta a') is a *single symbol*[7] and should be thought of as 'a little bit of a'. American texts often use Δa rather than δa for the same purpose.[8]

We observe that

$$(a + \delta a) \times (b + \delta b) = a \times b + \delta a \times b + \delta b \times a + \delta a \times \delta b.$$

The first term is much bigger than any of the other terms and so, looking only at this term, we obtain

$$(a + \delta a) \times (b + \delta b) \approx a \times b$$

'to the zeroth approximation'. The second and third terms are small compared with the first term, but large compared with the fourth term, so to obtain a more accurate result, we now look at the first three terms to obtain

$$(a + \delta a) \times (b + \delta b) \approx a \times b + \delta a \times b + \delta b \times a.$$

'to the first approximation'.

Looking at the algebra, we see that our argument does not depend on the sign of the various terms, but simply on their magnitude.

Exercise 1.2.1. *Use this insight to find*

$$20\,019 \times 299\,987$$

'to the zeroth and first order'. Find the result exactly for the purposes of comparison.

We can push things a little further. One obvious direction is to try multiplying three terms $a + \delta a$, $b + \delta b$ and $c + \delta c$, where δa, δb and δc are very small

[7] In particular, it does not mean $\delta \times a$.
[8] In the Greek alphabet, Δ is the capital version of δ so, conveniently, Δa and δa have the same pronunciation.

compared to a, b and c. We observe that

$$\begin{aligned}&(a+\delta a)\times(b+\delta b)\times(c+\delta c)\\&\quad = a\times b\times c+\delta a\times b\times c+\delta b\times a\times c+\delta c\\&\qquad \times b\times a+\delta a\times\delta b\times c+\delta a\times\delta c\times b+\delta b\times\delta c\times a+\delta a\times\delta b\times\delta c.\end{aligned}$$

The first term is much bigger than any of the other terms and so, looking only at this term, we obtain

$$(a+\delta a)\times(b+\delta b)\times(c+\delta c)\approx a\times b\times c$$

'to the zeroth order'. The next three terms are small compared with the first term, but large compared with the remaining terms, so, to obtain a more accurate result, we now look at the first three terms to obtain

$$\begin{aligned}(a+\delta a)\times(b+\delta b)\times(c+\delta c)\approx a\times b\times c+\delta a\times b\times c+\delta b\times a\times c\\+\delta c\times b\times a\end{aligned}$$

'to the first order'.

It is now natural to observe that the next three terms are small compared to the previous terms, but large compared to the final term, so

$$\begin{aligned}&(a+\delta a)\times(b+\delta b)\times(c+\delta c)\\&\quad\approx a\times b\times c+\delta a\times b\times c+\delta b\times a\times c+\delta c\\&\qquad\times b\times a+\delta a\times\delta b\times c+\delta a\times\delta c\times b+\delta b\times\delta c\times a\end{aligned}$$

'to the second order'. However, for the moment, we shall confine ourselves to zeroth order and first order approximations.[9]

Exercise 1.2.2. *Suppose that δx is small in magnitude compared with x. Find $(x+\delta x)^3$ to the first order.*

Suppose that n is a positive integer. Find $(x+\delta x)^n$ to the first order, giving reasons for your answer.

What happens if we consider other operations? Addition is very simple, since

$$(a+\delta a)+(b+\delta b)=a+b+(\delta a+\delta b),$$

so we have the unremarkable observation that

$$(a+\delta a)+(b+\delta b)\approx a+b$$

[9] At some point, the reader should observe that two zeroth order estimates of the same quantity may differ to the first order, two first order estimates of the same quantity may differ to the second order and so on.

to the zeroth order and

$$(a + \delta a) + (b + \delta b) \approx a + b + (\delta a + \delta b),$$

to the first order.

Exercise 1.2.3. *Make a similar observation for* $(a + \delta a) - (b + \delta b)$.

What about

$$\frac{b + \delta b}{a + \delta a}\ ?$$

Note first that we know how to deal with multiplication, so it suffices to look at the simpler problem of evaluating

$$\frac{1}{a + \delta a}$$

approximately. If we stare at the problem long enough, the following idea may occur to us. Let us write

$$u = \frac{1}{a} \text{ and } u + \delta u = \frac{1}{a + \delta a}.$$

If δa is small in magnitude compared to a, then $a + \delta a$ will be close to a and so $u + \delta u$ will be close to u. In other words, δu will be small in magnitude compared to u. Thus, working to first order,

$$\begin{aligned} 1 &= (a + \delta a) \times (u + \delta u) \approx a \times u + a \times \delta u + u \times \delta a \\ &= 1 + a \times \delta u + u \times \delta a. \end{aligned}$$

Subtracting 1 from both sides and rearranging, we get

$$a \times \delta u \approx -u \times \delta a,$$

so

$$a \times \delta u \approx -\frac{1}{a} \times \delta a,$$

that is to say,

$$\delta u \approx -\frac{1}{a^2} \times \delta a$$

to the first order.

Combining the results of the last paragraph with our previous results on multiplication, we see that, if δa and δb are small in magnitude compared with

a and b,

$$\frac{b+\delta b}{a+\delta a} = (b+\delta b)\times(u+\delta u) \approx b\times u + b\times \delta u + u\times \delta b$$
$$\approx \frac{b}{a} - \frac{b}{a^2}\times \delta a + \frac{1}{a}\times \delta b$$

to the first order.

The same kind of idea can be used to estimate $\sqrt{a+\delta a}$ when a is positive and large in magnitude compared with δa. Let us write

$$v = \sqrt{a} \text{ and } v+\delta v = \sqrt{a+\delta a}.$$

If δa is small in magnitude compared to a, then $a+\delta a$ will be close to a and so $v+\delta v$ will be close to v. In other words, δv will be small in magnitude compared to v. Thus, working to first order,

$$a+\delta a = (v+\delta v)^2 \approx v^2 + 2v\times \delta v = a + 2\sqrt{a}\times \delta v.$$

Subtracting a from both sides and rearranging, we get

$$\delta a \approx 2\sqrt{a}\times \delta v,$$

so

$$\delta v \approx \frac{1}{2\sqrt{a}}\times \delta a$$

to the first order.

Exercise 1.2.4. *Estimate $\sqrt{1\,000\,003}$ and compare your estimate with the answer given by your calculator.*

Exercise 1.2.5. *We write $a^{1/3}$ for the cube root of a. Find a first order estimate for $(a+\delta a)^{1/3}$ when a is large in magnitude compared with δa. Estimate $1\,000\,003^{1/3}$ and compare your estimate with the answer given by your calculator.*

We can obtain many other results by combining the ones we already have.

Example 1.2.6. *Heron's formula states that the area A of a triangle whose sides have lengths a, b and c is given by*

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}.$$

Use the formula to estimate the difference in area between a triangle whose sides have lengths a, b and c and a triangle whose sides have lengths $a+\delta a$, $b+\delta b$ and $c+\delta c$, where, as usual, δa, δb and δc are small in magnitude compared with a, b and c.

Solution. Let us write $u = s - a$, $v = s - b$, $w = s - c$ and $T = suvw$ and let us denote the corresponding quantities for the new triangle by $u + \delta u$, $v + \delta v$, $w + \delta w$ and $T + \delta T$. The quantities $A + \delta A$ and $s + \delta s$ will have the obvious meanings.

Our earlier result on square roots tells us that

$$A + \delta A = \sqrt{T + \delta T} \approx \sqrt{T} + \frac{1}{2\sqrt{T}}\delta T = A + \frac{1}{2A}\delta T$$

so that

$$\delta A \approx \frac{1}{2A}\delta T$$

to the first order. A simple modification of our rule concerning products of two and three objects gives[10]

$$\begin{aligned} T + \delta T &= (s + \delta s)(u + \delta u)(v + \delta v)(w + \delta w) \\ &\approx suvw + uvw\delta s + svw\delta u + suw\delta v + suv\delta w \\ &= T + uvw\delta s + svw\delta u + suw\delta v + suv\delta w, \end{aligned}$$

so that

$$\delta T \approx uvw\delta s + svw\delta u + suw\delta v + suv\delta w$$

or, more neatly,

$$\delta T \approx T\left(\frac{\delta s}{s} + \frac{\delta u}{u} + \frac{\delta v}{v} + \frac{\delta w}{w}\right)$$

to the first order.

Combining our two results gives

$$\delta A \approx \frac{1}{2A} \times T\left(\frac{\delta s}{s} + \frac{\delta u}{u} + \frac{\delta v}{v} + \frac{\delta w}{w}\right) = \frac{\sqrt{A}}{2}\left(\frac{\delta s}{s} + \frac{\delta u}{u} + \frac{\delta v}{v} + \frac{\delta w}{w}\right)$$

to the first order. This result is nice and symmetric, but the reader may object that, for example, δs depends on δa, δb and δc. She is invited to do the next exercise.

Exercise 1.2.7. *We use the notation of the last example. Show that*

$$\delta u = \frac{\delta b + \delta c - \delta a}{2}$$

and write down a first order formula for δA in terms of a, b, c, δa, δb and δc.

[10] Note that, having established the convention that δa is a single symbol, we have reverted to the standard practice of writing $a \times b = ab$ and $x \times \delta y = x\delta y$.

I suspect that your formula will not be particularly pretty. Write down a more elegant first order formula for δA in terms of u, v, w, δu, δv and δw.

The reader may ask if the ideas of this section are useful or merely some kind of party trick. She should note that many engineers and physicists are wizards at this sort of calculation, which strongly suggests that they find it useful.

Engineers ask 'what happens if I tweak this process?'. Just as the formula we discussed shows what happens to the area of a triangle if we make a small change in the lengths of the sides, so engineers can make 'back of an envelope' calculations about what happens to an output when they make small changes in the inputs.

Physicists know that all measurements have errors. They ask how much those errors affect the final result. The formula just obtained shows how small errors in measurement in the lengths of the sides would change the calculated area and the same technique enables physicists to make 'back of an envelope' calculations about the cumulative effects of different errors in measurement.

It is clear that what I have presented here is more of an art than a science. As the reader learns more about the calculus, she will find that there are relatively simple rules (for example, the 'function of a function' rule which we meet later) that cover many of the circumstances when we wish to perform approximate calculations, but sometimes the only tool available will be her own unaided ingenuity.

A more basic problem is 'how small is small?'. How small do the 'delta quantities' have to be relative to the 'large quantities' for first order calculations to be useful? Mathematicians have developed heavy mathematical machinery (for example, Taylor's theorem, which we meet later) for the purpose of answering this question, but most physicists and engineers rely on *experience* rather than *theory* to decide 'how small the deltas need to be' and do so very successfully.[11]

Advice. In order to gain the experience required, cultivate the habit, whenever you meet an interesting formula, of calculating the effect of small changes in the variables to first order and then thinking how small the changes need to be for your calculation to be useful.

Notational remark. We have used the approximation symbol in approximate equations like

$$\delta x \approx \delta y \text{ to first order.}$$

[11] Any fool can build something which will last forever and cost a thousand dollars. An engineer can build something that lasts as long as is required and costs a hundred dollars.