CHAPTER 1

Introduction and Motivation

'Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare' (Myerson 1997, p.1). The underlying idea here is that the decisions of the concerned individuals, who behave rationally, will influence each other's interests/pay-offs. No single person alone can determine the outcome completely. Each person's success depends on the actions of the other concerned individuals as well his own actions. Thus, loosely speaking, game theory deals with the mathematical formulation of a decision-making problem in which the analysis of a competitive situation is developed to determine an optimal course of action for a set of concerned individuals. Aumann (1987; 2008) suggested the alternative term 'interactive decision theory' for this discipline. However, Binmore (1992) argued that a game is played in a situation where rational individuals interact with each other. For instance, price, output, etc. of a firm will be determined by its actions as a decision maker. Game theory here describes how the firm will frame its actions and how these actions will determine the values of the concerned variable. Likewise, when two or more firms collude to gain more power for controlling the market, it is a game.

To understand this more clearly, consider a set of firms in an oligopolistic industry producing a common output. Each firm must not only be concerned with how its own output affects the market price directly; it must also take into consideration how variations in its output will affect the price through its effect on the decisions taken by other firms. Thus, strategic behaviour becomes an essential ingredient of the analysis. A tool that economists employ for modelling this type of situation is non-cooperative game theory.

As a second example, consider a landowner who owns a large piece of land on which some peasants work. The landowner does not work and requires at least one peasant to work on the piece of land to produce some output. On the other hand, the peasants cannot produce anything on their own because they require land. This shows that cooperation between the landowner and peasants is necessary for production of some output, otherwise no production will occur. A situation of this type is modelled by coalition or cooperative game.

In the next section of this introductory chapter, we provide a brief historical sketch of the development of some of the important concepts in game theory. However, we do not claim to present a complete survey of the development in the subject. For further details, an interested reader can see, among others Fudenberg and Tirole (1991), Binmore (1992), Osborne and Rubinstein (1994), Owen (1995), Myerson (1997), Chatterjee and Samuelson (2001) and Peters (2008). We then present a brief introduction to the remaining chapters of the book.

1.1 A Brief Historical Sketch

There are two approaches to the theory of games: the strategic approach and coalitional or cooperative approach. A non- cooperative or procedural game specifies all the possible actions for each individual decision maker, generally referred to as a player. Each course of action open to a player is called a strategy. A strategy is called a pure strategy if it is chosen with certainty, whereas mixed strategy for a player is a probability distribution over his pure strategies.

Modern game theory began with John von Neumann's (1928) classical saddle point theorem for a two-person zero-sum game in which one player's gain is matched by the other player's loss. This was followed by the seminal book by John von Neumann and Oscar Morgenstern (1944), a culmination of rich collaboration between the authors. It provides an excellent treatment of many types of games along with extensive discussions on potential applications of game theory. The book builds up two notions of representation of non-cooperative games: the normal or strategic form and the extensive form. The former specifies the players' strategy sets and their pay-off functions. Each player chooses a strategy and the strategy combination chosen by the players determines a pay-off for each of the players. A game of this type is a single-period or one-shot game. The latter corresponds to presentation of the game in terms of the sequential actions of the players, that is, through a movement of the players and through specification of strategies adopted by each player. Thus, while the former does not take into account the temporal structure of a game, the latter incorporates it explicitly.

von Neumann and Morgenstern's pioneering contribution have inspired researchers to work extensively on the theory of games and its application issues. One such scientist who contributed significantly to both streams of game theory was John F. Nash. In 1951, he developed a general formulation of equilibrium in a non-cooperative game which is now popularly known as the Nash equilibrium. A combination of strategies of the players in a non-cooperative game for which each player maximizes his own pay-off with respect to his own strategy choice, given the strategy choices of the other players, is said to constitute a Nash equilibrium. Equivalently, under Nash equilibrium, holding the strategies of the other players fixed, no player can obtain a higher pay-off by choosing a different strategy. A refinement of the Nash equilibrium was developed by Selten (1975) under the name sub-game perfect equilibrium. A sub-game is a subset of a game which, when considered in isolation, is a game on its own. A strategy combination is called a sub-game perfect equilibrium if it constitutes a Nash equilibrium for each sub-game in the game. Aumann (1974; 1987a) introduced the concept of correlated equilibrium, which is more flexible than the Nash equilibrium. A correlated equilibrium allows statistical dependence among strategies of the players, which is not permissible under the Nash equilibrium. An extensive formal analysis of common knowledge assumption and its implications were investigated by Aumann (1976). A game is characterized by complete information if each player has full knowledge about the characteristics (strategies and pay-offs) of other players. For instance, in a perfectly competitive market, all sellers and buyers possess complete information about the price and quality of the product. John C. Harsanyi developed the concept of Bayesian games in 1967−68. In a Bayesian game, a player does not have complete information on other players' characteristics.

The famous prisoners' dilemma was introduced as an example of non-zero sum game in the 1950s. This dilemma is similar in nature to a situation in an oligopolistic industry where arrangements that benefit the firms in the industry when they act as a cartel, create high incentives for individual firms to deviate from the arrangement. If each firm follows individual interests by deviating from the cartel arrangement, then the arrangement falls apart. The excellent book, *Games and Economic Decisions*, by R.D. Luce and H. Raiffa, which was published in 1957, is one of the first references that provide extensive discussion on the prisoners' dilemma.

The application of game theory to biology is dealt with in John Maynard Smith's book, *Evolution and the Theory of Games*, published in 1982. The main focus was on evolutionary stable strategy, a strategy, which when adopted by all members of a population, over evolutionary time, can withstand any alternative mutant strategy. An evolutionary stable strategy is an example of gradual cooperation representing a Nash equilibrium. Schelling (1960) worked on early examples of gradual cooperation. His famous book, *The Strategy of Conflict*, is regarded as a classical contribution to the understanding of issues like conflict, commitment, and coordination.

Hurwicz (1972; 1973) initiated the mechanism design theory, which was further developed by Maskin (1999) and Myerson (1979; 1981). By a mechanism, we mean a communication system through which players exchange their messages with each other and the messages that together influence the determination of the outcome. An important characteristic, which was formalized by Hurwicz in this context, was incentive compatibility, a requirement which demands that each player knows that his best strategy is guided by the rules, irrespective of what others decide to do.

von Neumann and Morgenstern (1944) considered cooperative games for several players.

> Cooperative theory starts out with a formalization of games (the coalition form) that abstracts away altogether from procedures and ... concentrates, instead, on the possibilities of agreement.... There are several reasons that cooperative game came to be treated separately. One is that when one does build negotiation and enforcement procedures explicitly into the model, then the results of a non-cooperative analysis depend very strongly on the precise form of the procedures, on the order of making offers and counter-offers, and so on. This may be appropriate in voting situations in which precise rules of parliamentary order prevail, where a good strategist can indeed carry the day. But problems of negotiation are usually more amorphous, it is difficult to pin down just what the procedures are. More fundamentally, there is a feeling that procedures are not really all that relevant; that it is the possibilities for

> coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak. ... Detail distracts attention from essentials. Some things are seen better from a distance; the Roman camps around Metzada are indiscernible when one is in them, but easily visible from the top of the mountain. (Aumann 1987, p. 463)

Aumann's (1987) argument clearly indicates that essential to the notion of cooperative game is coalition formation. von Neumann and Morgenstern (1944) deal with the patterns of coalition formation under rational behaviour of the players. A coalition is simply a subset of the player set. Cooperative game theory deals with situations where the objectives of the participants of the game are partially cooperative and partially conflicting. It is in the interest of the participants to cooperate, in the sense of making binding agreements, for achieving the maximum possible benefit. When it comes to the distribution of benefit/pay-offs, participants have conflicting interests. Such situations are usually modelled as cooperative games. There is complete information on rules of the game, all available strategies and pay-offs in all possible situations. Participants are free to cooperate, negotiate, bargain, collude, make binding agreements with one another, form coalitions or subgroups, make threats and even withdraw from a coalition. Any subgroup of players can make contractual agreements independently of the remaining players. Therefore, cooperative game theory looks for possible sets of outcomes, investigates what the participants can achieve, which coalitions will form, how the benefits will be divided among the members of a coalition and to what extent the outcomes will be stable.

We may illustrate the situation by an example. Consider a society in which each individual is endowed with a bundle of goods that can be used as inputs in a production process. All production processes are assumed to produce the same output which can be distributed among the individuals. Assume also that the inputs are complementary. Then in order to maximize total output, individuals may need to exchange inputs. This is where cooperation arises. When the problem of distribution of benefits of the cooperation arises, there may be a conflict of individual interests. In other words, the individuals would like to investigate whether there are incentives to cooperate and how to allocate the benefits of cooperation among themselves. In order to resolve the problem, a game theoretic analysis may be quite appropriate. Such a game is called a market game.

Nash (1950) suggested a two-person fixed threat bargaining model using an axiomatic approach for this problem. In this two-person cooperative game, each player obtains a fixed pay-off if the agreement between the players fails. There is a feasible set of outcomes that the players can achieve if they succeed in making an agreement. However, in the absence of an agreement, no player can help or hurt himself or his partner. The unique outcome of the bargaining game is the element of the set of attainable pay-off pairs that maximizes the product of gains from the agreement.

Two pioneering contributions that form important basis in cooperative game theory were from Shapley (1953) and Gillies (1959). Gillies (1959) suggested that the core of cooperative games can be a general solution concept. Shapley (1953) introduced what is known as the 'Shapley value' as a further solution concept. While the core consists of a set of possible allocations satisfying certain conditions, the Shapley value establishes a unique allocation with specific properties. Aumann and Maschler (1964) suggested an alternative method, which has been referred to as the bargaining set. The trend to develop more and more solution concepts possessing varying properties is still an important topic of research in cooperative game theory. We provide an intuitive discussion on several important solution concepts in the next section. More elaborate discussions are presented in subsequent chapters.

We will discuss the history of each solution concept of cooperative games in the corresponding chapter. In order to avoid repetition, we did not discuss it in detail in this chapter. However, the history of non-cooperative games has been presented in detail simply because we do not proceed further with this concept and our presentation will give the reader an idea about the development of the subject.

1.2 An Overview of the Chapters

Cooperative games are divided into two categories: games with transferable utilities and games with non-transferable utilities. By a cooperative game with transferable utilities, we mean a game in which the opportunities available to each coalition is represented by a single number, such as money, interpreted as the pay-off or utility available to the coalition. The members of the coalition are free to divide this amount among themselves in a mutually agreeable manner. That is, the result of cooperation can be numerically quantified and transferred among the members of the coalition involved in the cooperation without any loss or gain. For instance, if the players in a game are firms in a market and utilities/profits of the coalitions are measured in terms of money, then the underlying profit division game is a transferable utility game. Transferability greatly simplifies the analysis. It enables us to define the characteristic function which specifies the worth of any arbitrary coalition. In a non-transferable utility game, the opportunities available at the disposal of a coalition may be represented by a set of vectors rather than by a single number. To understand this, consider an exchange economy consisting of $n \geq 2$ agents. Each agent has an initial endowment of $k \geq 2$ commodities and a preference relation defined on the set of allocation X of k goods to n agents. The initial endowment is an allocation that shows the amount of each good that the consumers bring to the market for exchange. In an exchange economy, the agents, through exchange of their endowments, try to determine some mutually advantageous trade. For any coalition of agents, the value set consists of those elements of X such that the total amount of each good allocated to the members of the coalition equals the total amount of their initial endowments of the good. The agents outside the coalition do not participate in any trading and hold on their initial endowments. For all the agents as a whole, the value set is the set of all feasible allocations. Given prices for different goods, an allocation in this pure exchange economy is a competitive equilibrium if it maximizes the preference for each individual subject to his budget constraint and all the choices are consistent in the sense that equality must hold between total demand and total supply for each good. The first fundamental theorem of Welfare Economics asserts that a competitive equilibrium is in the (Edgeworth) core and hence, it is Pareto efficient. The Edgeworth core is the set of all feasible allocations that cannot be improved upon by any coalition of individuals. An allocation of the fixed quantities of goods in an exchange economy is Pareto efficient if through reallocation of goods no individual can be made better off without making at least one individual worse off. We may refer to trading of goods in this economy as an exchange economy game. In this case, we are not comparing utilities of two agents or transferring utility, and hence this is an example of non-transferable utility game. A two-person bargaining game is also an example of a cooperative game with non-transferable utility. In this book, we will be mainly concerned with transferable utility games. For a substantial discussion on non-transferable utility games, the interested reader can refer to Peleg and Sudhölter (2007).

In cooperative game theory with transferable utility, the pay-off function open to each coalition is described by a characteristic function which associates with each coalition the total utility that the members of the coalition can achieve when they work in concert. Thus, in this case, the focus is on coalitions and their pay-offs. For any coalition, the utility that the characteristic function assigns to it is known as the worth that the coalition can achieve when its members act together. In other words, the worth of a coalition is the amount that the members of the coalition can earn on their own.

We illustrate the concept of characteristic function by giving an example. Suppose person A has an old car to sell which is worthless to him unless he can sell it. Person B, a prospective buyer, values the car at USD 1000, while person C, a second buyer, values it at USD 1050. The game consists of each of the two prospective buyers pricing the car, and the seller accepting the higher price or rejecting both. The general idea here is that by transferring ownership of the car from the seller to one of the buyers, utility is created. The set of coalitions of this 3-person player set is $\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}, \emptyset\}$, where $\emptyset$ is the empty set. Thus, coalition $\{A, B\}$ can create 1000 units (dollars) of utility, which the two players can divide between them in any way they choose. For instance, if they decide on a price of USD 550, then person A gains 550 units (he has exchanged a worthless car), while person B gains 450 units (he has obtained a car which he considers of worth USD 1000 for USD 550). Likewise, coalition $\{A, C\}$ can derive 1050 units of utility. The three single player coalitions, $\{A\}$, $\{B\}$ and $\{C\}$, and the coalition $\{B, C\}$ do not obtain any utility because in these cases there is no interaction. Moreover, if the coalition $\{A, B, C\}$ is formed, the best option would be to sell the car to person C and derive a total of 1050 units of utility. Since the empty set does not contain any player, it is a convention that this set creates zero utility. If we denote the characteristic function by v, then we have: $v(\{A\}) = v(\{B\}) = v(\{C\}) = v(\{B, C\}) = 0$, $v(\{A, B\}) = 1000$, $v(\{A, C\}) = v(\{A, B, C\}) = 1050$, and $v(\emptyset) = 0$. In general, for an n-person coalitional (or characteristic function form) game with player set $N = \{A_1, A_2, \ldots, A_n\}$, the characteristic function v is a real valued function defined on the set of all coalitions (subsets of N) satisfying $v(\emptyset) = 0$. The number of possible coalitions here is 2^n.

To illustrate the idea of cooperation further, consider a situation in which there are some potential users of a public service. The cost function determines the cost of providing the service to any group of users in the most efficient way in terms of minimum cost. Cooperation among the service provider and the users, that is, the service provider asks for a payment and the users in exchange of the service agree to make some payment, will ensure the efficient way of serving the users. Consider another example: suppose there are two sellers and one buyer of an indivisible good. Each seller offers to sell one unit of the good and can make the product available at a particular price. The buyer sets a worth on the product and is interested in paying the lowest possible price. He does not want to pay more than his worth valuation. This buyer−seller interaction problem can be modelled as a coalition form game.

Suppose a game is played. A natural question from the players would be how to determine the pay-offs expected from their participation in the game. This is not an easy question to answer and the characteristic function, which determines the joint pay-off of the members of a coalition, does not provide a solution to it. If there are two players in a game, each essentially faces a yes−no question, to cooperate or not to cooperate. However, if there are more than two players, the situation may change substantially. To understand this, let us consider a profit-sharing game in which five persons A_1, A_2, A_3, A_4 and A_5 are partners of a joint business. In this game, there can be $2^5 = 32$ coalitions. Table 1.1, which is taken from Curiel (1997), presents the worth of each possible coalition.

In the above game, each of the partners has contributed some capital and skill to the joint venture. The partners are required to divide an annual profit of USD 100 from the joint venture among them. Therefore, the grand coalition $N = \{A_1, A_2, A_3, A_4, A_5\}$ as a whole earns USD 100. At the outset, the trivial solution of assigning USD 20 to each partner appears to be sensible. However, after a careful analysis, A_4 and A_5 observe that if they work jointly without the other three, they can make a profit of USD 45. Therefore, the equal division allocation will not be acceptable to the coalition $\{A_4, A_5\}$. It turns out that a coalition by A_1, A_2 and A_3 can earn a joint profit of USD 25 only. Hence, these three persons will be quite keen to keep A_4 and A_5 in their coalition. They can decide to give an amount higher than USD 45 (say, USD 46) to A_4 and A_5, and divide the remaining USD 54 equally among themselves. Although this looks like a solution to the problem, after a further analysis, A_3, A_4 and A_5 observe that they can make a joint profit of USD 70. This profit is higher than USD 64 (USD 46+USD 18), which is assigned to them under the second allocation. It, therefore, rules out the possibility of acceptance of the second allocation to the coalition $\{A_3, A_4, A_5\}$. A_1 and A_2 can agree to give USD 70 to A_3, A_4 and A_5, and divide the remaining USD 30 between them. However, if the coalition $\{A_2, A_4, A_5\}$ is formed, its aggregate profit becomes USD 65, which is higher than USD $(2 \times \frac{70}{3} + \frac{30}{2})$, the amount it receives under the last allocation. This implies that A_2, A_4 and A_5 will not be satisfied with this allocation.

Table 1.1 Profit-sharing game

Coalition : S	*Worth* : $v(S)$	*Coalition* : S	*Worth* : $v(S)$
$\{A_1\}$	0	$\{A_1, A_2, A_5\}$	40
$\{A_2\}$	0	$\{A_1, A_3, A_4\}$	40
$\{A_3\}$	0	$\{A_1, A_3, A_5\}$	45
$\{A_4\}$	5	$\{A_1, A_4, A_5\}$	55
$\{A_5\}$	10	$\{A_2, A_3, A_4\}$	50
$\{A_1, A_2\}$	0	$\{A_2, A_3, A_5\}$	55
$\{A_1, A_3\}$	5	$\{A_2, A_4, A_5\}$	65
$\{A_1, A_4\}$	15	$\{A_3, A_4, A_5\}$	70
$\{A_1, A_5\}$	20	$\{A_1, A_2, A_3, A_4\}$	60
$\{A_2, A_3\}$	15	$\{A_1, A_2, A_3, A_5\}$	65
$\{A_2, A_4\}$	25	$\{A_1, A_2, A_4, A_5\}$	75
$\{A_2, A_5\}$	30	$\{A_1, A_3, A_4, A_5\}$	80
$\{A_3, A_4\}$	30	$\{A_2, A_3, A_4, A_5\}$	90
$\{A_3, A_5\}$	35	N	100
$\{A_4, A_5\}$	45	$\emptyset$	0
$\{A_1, A_2, A_3\}$	25		
$\{A_1, A_2, A_4\}$	35		

This example clearly indicates that in order to make commitments about the contribution of skill and capital, the partners in a possible coalition will require prior information on the division of profit. A systematic analysis concerning profit division which will clearly consider how much each coalition can acquire, is necessary. The focus of interest will be the partners' bargaining power over the division of profit. A player has to decide which of the many possible coalitions to join. He will also have to take into account the extent to which players outside his coalition will coordinate their actions. Loosely speaking, 'who needs whom more?' For instance, if A_2 forms coalitions with A_4 and A_5