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Performance Analysis of Complex Networks and Systems

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**Waar een wil is, is een weg.
to my father**

**in memory of
my wife Saskia and my son Nathan**

to my sons Vincent and Laurens

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Frontmatter

[More information](#)

Contents

<i>Preface</i>	xi
<i>Symbols</i>	xv
1 Introduction	1
Part I Probability theory	5
2 Random variables	7
2.1 Probability theory and set theory	7
2.2 Discrete random variables	12
2.3 Continuous random variables	15
2.4 The conditional probability	22
2.5 Several random variables and independence	24
2.6 Conditional expectation	32
2.7 Problems	33
3 Basic distributions	35
3.1 Discrete random variables	35
3.2 Continuous random variables	40
3.3 Derived distributions	44
3.4 Functions of random variables	54
3.5 Examples of other distributions	57
3.6 Summary tables of probability distributions	64
3.7 Problems	65
4 Correlation	69
4.1 Generation of correlated Gaussian random variables	69
4.2 Generation of correlated random variables	76
4.3 The non-linear transformation method	77

vi	<i>Contents</i>	
	4.4	Examples of the non-linear transformation method 82
	4.5	Linear combination of independent auxiliary random variables 86
	4.6	Sampling and estimators 90
	4.7	Problems 98
5		Inequalities 99
	5.1	The minimum (maximum) and infimum (supremum) 100
	5.2	Continuous convex functions 100
	5.3	Inequalities deduced from the Mean Value Theorem 102
	5.4	The Markov, Chebyshev and Chernoff inequalities 103
	5.5	The Hölder, Minkowski and Young inequalities 106
	5.6	The Gauss inequality 110
	5.7	The dominant pole approximation and large deviations 112
	5.8	Problems 114
6		Limit laws 115
	6.1	General theorems from analysis 115
	6.2	Law of Large Numbers 118
	6.3	Central Limit Theorem 120
	6.4	The Law of Proportionate Effect 121
	6.5	Logarithm of a sum of random variables 123
	6.6	Extremal distributions 126
	6.7	Problem 132
		Part II Stochastic processes 133
7		The Poisson process 135
	7.1	A stochastic process 135
	7.2	The Poisson process 139
	7.3	Properties of the Poisson process 141
	7.4	The Poisson process and the uniform distribution 146
	7.5	The non-homogeneous Poisson process 150
	7.6	The failure rate function 152
	7.7	Problems 154
8		Renewal theory 157
	8.1	Basic notions 158
	8.2	Limit theorems 164
	8.3	The residual waiting time 169
	8.4	The renewal reward process 173
	8.5	Problems 176

Contents

vii

9	Discrete-time Markov chains	179
9.1	Definition	179
9.2	Discrete-time Markov chains	180
9.3	The steady-state of a Markov chain	191
9.4	Example: the two-state Markov chain	197
9.5	A generating function approach	199
9.6	Problems	201
10	Continuous-time Markov chains	205
10.1	Definition	205
10.2	Properties of continuous-time Markov processes	206
10.3	Steady-state	212
10.4	The embedded Markov chain	214
10.5	The transitions in a continuous-time Markov chain	219
10.6	Example: the two-state Markov chain in continuous-time	220
10.7	Time reversibility	221
10.8	Problems	224
11	Applications of Markov chains	227
11.1	Discrete Markov chains and independent random variables	227
11.2	The general random walk	228
11.3	Birth and death process	234
11.4	Slotted Aloha	247
11.5	Ranking of webpages	251
11.6	Problems	255
12	Branching processes	257
12.1	The probability generating function	258
12.2	The limit W of the scaled random variables W_k	262
12.3	The probability of extinction of a branching process	266
12.4	Conditioning of a supercritical branching process	269
12.5	Asymptotic behavior of W	273
12.6	A geometric branching process	276
12.7	Time-dependent branching process	278
12.8	Problems	285
13	General queueing theory	287
13.1	A queueing system	287
13.2	The waiting process: Lindley's approach	291
13.3	The Beneš approach to the unfinished work	295
13.4	The counting process	302
13.5	Queue observations	304

viii	<i>Contents</i>	
	13.6 PASTA	306
	13.7 Little's Law	306
14	Queueing models	311
	14.1 The M/M/1 queue	311
	14.2 Variants of the M/M/1 queue	316
	14.3 The M/G/1 queue	322
	14.4 The GI/D/m queue	327
	14.5 The M/D/1/K queue	334
	14.6 The G/M/1 queue	337
	14.7 The N*D/D/1 queue	341
	14.8 The AMS queue	345
	14.9 The cell loss ratio	349
	14.10 Problems	353
	Part III Network science	357
15	General characteristics of graphs	359
	15.1 Introduction	359
	15.2 The number of paths with j hops	361
	15.3 The degree of a node in a graph	362
	15.4 The origin of power-law degree distributions in complex networks	364
	15.5 Connectivity and robustness	366
	15.6 Graph metrics	368
	15.7 Random graphs	375
	15.8 Interdependent networks	389
	15.9 Problems	393
16	The shortest path problem	397
	16.1 The shortest path and the link weight structure	398
	16.2 The shortest path tree in K_N with exponential link weights	399
	16.3 The hopcount h_N in the URT	404
	16.4 The weight of the shortest path	409
	16.5 Joint distribution of the hopcount and the weight	412
	16.6 The flooding time T_N	414
	16.7 The degree of a node in the URT	418
	16.8 The hopcount in a large, sparse graph	423
	16.9 The minimum spanning tree	431
	16.10 Problems	438
17	Epidemics in networks	443

Contents

ix

17.1	Classical epidemiology	444
17.2	The continuous-time SIS Markov process	446
17.3	The governing ε -SIS equations	450
17.4	N -Intertwined Mean-Field Approximation (NIMFA)	462
17.5	Heterogeneous N -intertwined mean-field approximation	471
17.6	Epidemics on the complete graph K_N	473
17.7	Non-Markovian SIS epidemics	477
17.8	Heterogeneous mean-field (HMF) approximation	484
17.9	Problems	486
18	The efficiency of multicast	489
18.1	General results for $g_N(m)$	490
18.2	The random graph $G_p(N)$	493
18.3	The k -ary tree	501
18.4	The Chuang–Sirbu Law	503
18.5	Stability of a multicast shortest path tree	506
18.6	Proof of (18.16): $g_N(m)$ for random graphs	509
18.7	Proof of Theorem 18.3.1: $g_N(m)$ for k -ary trees	513
18.8	Problems	515
19	The hopcount and weight to an anycast group	517
19.1	Introduction	517
19.2	General analysis of the hopcount	520
19.3	The k -ary tree	523
19.4	The uniform recursive tree (URT)	523
19.5	The performance measure η in exponentially growing trees	531
19.6	The weight to an anycast group	533
<i>Appendix A</i>	A summary of matrix theory	539
<i>Appendix B</i>	Solutions of problems	575
<i>References</i>		663
<i>Index</i>		673

Cambridge University Press

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Piet Van Mieghem

Frontmatter

[More information](#)

Preface

This book is a development of *Performance Analysis of Communications Networks and Systems* of 2006. Its current incarnation has a broader scope, extending to complex networks, the broad term covering all types of real-world networks, that encompass communications networks. Apart from the correction of numerous errors in the earlier book, nearly all chapters have been extended. Chapter 17 on Epidemics in networks has been added, while the appendix on algebraic graph theory has been deleted, because our book *Graph Spectra for Complex Networks* amply replaces this chapter. Similar to *Graph Spectra for Complex Networks*, **art. x** has been used to refer to article x in Appendix A. The number of problems, together with their solutions in Appendix B, has been doubled at least.

Performance analysis belongs to the domain of applied mathematics. In particular, the branches of mathematics as probability theory, stochastic processes and graph theory are exploited, besides analysis (calculus) and linear algebra that are omnipresent nearly everywhere. The major aim of this book is to offer several mathematical methods to address challenges in *network science*, the rapidly growing field of complex networking. The link with technology is kept shallow, on purpose, because most technical advances in micro-electronics, in communications protocols and services, standards, etc. have a more limited lifetime and a narrower scope compared to mathematical concepts.

This book aims to present methods rigorously, hence mathematically, with minimal resorting to intuition. It is my belief that intuition is often gained after the result is known and rarely before the problem is solved, unless the problem is simple. I have tried to interpret most of the important formulas in the sense of “What does this mathematical expression teach me?” This last step justifies the word “applied”, since most mathematical treatises do not interpret as it contains the risk to be imprecise and incomplete.

As prerequisites, familiarity with elementary probability and the knowledge of the theory of functions of a complex variable are assumed. In particular, the beautiful book on the *Theory of Functions* by Titchmarsh (1964) is recommended for complex function theory. Appendix A briefly summarizes concepts of linear algebra used in this book. Parts in the text in small font refer to more advanced topics or to computations that can be skipped at first reading.

Chapters 2–3 in Part I briefly review probability theory; they are included to make the remainder self-contained. Chapter 4 discusses how to compute correlation between several random variables. The central role of the Gaussian distribution is emphasized. Since the Gaussian distribution is so widely known, I have included Gauss’s own derivation of his distribution, whereas the Central Limit Theorem (studied in Chapter 6 together with other limit laws) is currently considered as the

main law that produces a Gaussian distribution. Chapter 5 treats powerful inequalities at an introductory level, but skips the recent (and more difficult) inequalities, such as the FKG inequality due to Fortuin, Kasteleyn and Ginibre.

The book essentially starts with Chapter 7 (Part II) on Poisson processes. The Poisson process (independent increments and discontinuous sample paths) and Brownian motion (independent increments but continuous sample paths) are considered to be the most important basic stochastic processes. We briefly touch upon renewal theory to move to Markov processes. The theory of Markov processes is regarded as a fundament for many applications, particularly in queueing theory, epidemics on networks and some instances of shortest path routing. A large part of the book is consumed by Markov processes and its applications. The last chapters of Part II dive into queueing theory. Inspired by intriguing problems in telephony at the beginning of the twentieth century, Erlang has pushed queueing theory to the scene of sciences. Since his investigations, queueing theory has grown considerably. Especially during the last decade with the advent of the Asynchronous Transfer Mode (ATM) and the worldwide Internet, many early ideas have been refined (e.g. discrete-time queueing theory, large deviation theory, scheduling control of prioritized flows of packets) and new concepts (self-similar or fractal processes) have been proposed. Queueing theory is expected to emerge again in complex networks, when structural aspects are understood.

Part III covers the parts of network science that study the structure of and processes on complex networks. While the first decade of this century has predominantly focused on the topology and structure of complex networks, the next decade aims to understand the influence of the topology on the dynamic process(es) on the network. Chapter 15 overviews the topological properties of complex networks and invokes concepts of graph theory, as well as random graph theory. Both the famous Erdős-Rényi and the Barabási-Albert random graphs are introduced. The giant component of a network, which is the largest still connected subgraph, is studied, because the giant component can be considered as the operational heart of the complex network. Finally, interdependent networks are introduced and the surprisingly different nature of cascading failures between single and interdependent networks is discussed. Chapter 16 and 17 exemplify two dynamic processes on a network: the transport along the shortest path and the spread of epidemics in networks. Chapter 18 and 19 dive deeper into routing instances as multicasting and anycasting, originally proposed for internetworking, but extendable to the newer communications types as social networking and cloud computing.

Since network science is still developing at fast pace, Part III is undoubtedly the least mature and complete. Moreover, I have predominantly relied on my own research for the exposition of topics, without the ambition to thoroughly review (and cite) contributions in the field. The book of Remco van der Hofstad (2013) treats random graphs differently inspired by Erdős' probabilistic method. Remco seriously extends the material in Chapter 15 and I recommend his book for a deeper

Cambridge University Press

978-1-107-05860-6 - Performance Analysis of Complex Networks and Systems

Piet Van Mieghem

Frontmatter

[More information](#)*Preface*

xiii

discussion about the giant component, phase transitions, inhomogeneous random graphs, the configuration model and preferential attachment models.

To Huijuan Wang, who has used the earlier book of 2006 in her classes, I am indebted for many suggestions, corrections and a shorter proof of the degree distribution in the URT in Section 16.7.2. Numerous people have pointed me to errors in the earlier 2006 book, while others gave suggestions for this book. I am very grateful to all of them: Chandrashekar Pataguppe Suryanarayan Bhat, Ruud van de Bovenkamp, Eric Cator, Li Cong, Edwin van Dam, Michel Dekking, David Hemley, Remco van der Hofstad, Gerard Hooghiemstra, David Hunter, Geurt Jongbloed, Merkouris Karaliopoulos, Rob Kooij, Javier Martin Hernandez, Jil Meier, Raphi Rom, Annalisa Socievole, Bart Steyaert, Siyu Tang, Stojan Trajanovski, Matthias Waehlich, Huijuan Wang, and to the many students at Delft University of Technology that followed my course on Performance Analysis over the last 14 years.

Although this book is intended to be of practical use, in the course of writing it, I became more and more persuaded that mathematical rigor has ample virtues of its own.

*Omnia sunt incerta, cum a
mathematicae discessum est*

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Frontmatter

[More information](#)

Symbols

Only when explicitly mentioned will we deviate from the standard notation and symbols outlined here.

Random variables and matrices are written with capital letters, while complex, real, integer, etc., variables are in lower case. For example, X refers to a random variable, A to a matrix, whereas x is a real number and z is complex number. Usually, i, j, k, l, m, n are integers. Operations on random variables are denoted by $[\cdot]$, whereas (\cdot) is used for real or complex variables. A set of elements is embraced by $\{\cdot\}$.

Linear algebra

A $n \times m$ matrix $\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$

$\det A$ determinant of a square matrix A ; also denoted by $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$

$\text{trace}(A)$ $= \sum_{j=1}^n a_{jj}$: sum of diagonal elements of A

$\text{diag}(a_k)$ $= \text{diag}(a_1, a_2, \dots, a_n)$: a diagonal matrix with diagonal elements listed, while all off-diagonal elements are zero

A^T transpose of a matrix, the rows of A are the columns of A^T

A^* matrix in which each element is the complex conjugate of the corresponding element in A

A^H $= (A^*)^T$: Hermitian of matrix A

$c_A(x)$ $= \det(A - xI)$: characteristic polynomial of A

$\text{adj}A$ $= A^{-1} \det A$: adjugate of A

$Q(\lambda)$ $= \frac{c_A(\lambda)}{\lambda I - A}$: adjoint of A

e_j basic vector, all components are zero, except for component j that is 1

δ_{kj} Kronecker delta, $\delta_{kj} = 1$ if $k = j$, else $\delta_{kj} = 0$

Probability theory

$\Pr[X]$	probability of the event X
$E[X]$	$= \mu$: expectation of the random variable X
$\text{Var}[X]$	$= \sigma_X^2$: variance of the random variable X
$f_X(x)$	$= \frac{dF_X(x)}{dx}$: probability density function of X
$F_X(x)$	probability distribution function of X
$\varphi_X(z)$	probability generating function of X
	$\varphi_X(z) = E[z^X]$ when X is a discrete r.v.
	$\varphi_X(z) = E[e^{-zX}]$ when X is a continuous r.v.
$\{X_k\}_{1 \leq k \leq m}$	$= \{X_1, X_2, \dots, X_m\}$
$X_{(k)}$	k -th order statistics, k -th smallest value in the set $\{X_k\}_{1 \leq k \leq m}$
P	transition probability matrix (Markov process)
$1_{\{x\}}$	indicator function: $1_{\{x\}} = 1$ if the event or condition $\{x\}$ is true, else $1_{\{x\}} = 0$. For example, $\delta_{kj} = 1_{\{k=j\}}$
γ	$= 0.577\ 215\dots$: Euler's constant
Ω	sample space
ω	sample point

Queuing theory

t_n	arrival time of the n -th packet
r_n	departure time of the n -th packet
$\tau_n = t_n - t_{n-1}$	n -th interarrival time
x_n	service time of n -th packet
w_n	waiting time of the n -th packet
$T_n = x_n + w_n$	system time of n -th packet
$v(t)$	virtual waiting time or unfinished work at time t
$\lambda = (E[\tau])^{-1}$	average arrival rate
$\mu = (E[x])^{-1}$	average service rate
$\rho = \frac{\lambda}{\mu}$	traffic intensity
$N_A(t)$	number of arrivals at time t
$N_S(t)$	number of packets in the system (queue plus server) at time t
$N_Q(t)$	number of packets in the queue at time t