

An Introduction to Sparse Stochastic Processes

Providing a novel approach to sparsity, this comprehensive book presents the theory of stochastic processes that are ruled by linear stochastic differential equations and that admit a parsimonious representation in a matched wavelet-like basis.

Two key themes are the statistical property of infinite divisibility, which leads to two distinct types of behavior – Gaussian and sparse – and the structural link between linear stochastic processes and spline functions, which is exploited to simplify the mathematical analysis. The core of the book is devoted to investigating sparse processes, including a complete description of their transform-domain statistics. The final part develops practical signal-processing algorithms that are based on these models, with special emphasis on biomedical image reconstruction.

This is an ideal reference for graduate students and researchers with an interest in signal/image processing, compressed sensing, approximation theory, machine learning, or statistics.

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“Over the last twenty years, sparse representation of images and signals became a very important topic in many applications, ranging from data compression, to biological vision, to medical imaging. The book *Sparse Stochastic Processes* by Unser and Tafti is the first work to systematically build a coherent framework for non-Gaussian processes with sparse representations by wavelets. Traditional concepts such as Karhunen-Lo  ve analysis of Gaussian processes are nicely complemented by the wavelet analysis of Levy Processes which is constructed here. The framework presented here has a classical feel while accommodating the innovative impulses driving research in sparsity. The book is extremely systematic and at the same time clear and accessible, and can be recommended both to engineers interested in foundations and to mathematicians interested in applications.”

David Donoho, *Stanford University*

“This is a fascinating book that connects the classical theory of generalised functions (distributions) to the modern sparsity-based view on signal processing, as well as stochastic processes. Some of the early motivations given by I. Gelfand on the importance of generalised functions came from physics and, indeed, signal processing and sampling. However, this is probably the first book that successfully links the more abstract theory with modern signal processing. A great strength of the monograph is that it considers both the continuous and the discrete model. It will be of interest to mathematicians and engineers having appreciations of mathematical and stochastic views of signal processing.”

Anders Hansen, *University of Cambridge*

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MICHAEL UNSER and POUYA D. TAFTI

École Polytechnique Fédérale de Lausanne



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Contents

	<i>Preface</i>	<i>page</i> xiii
	<i>Notation</i>	xv
1	Introduction	1
	1.1 Sparsity: Occam’s razor of modern signal processing?	1
	1.2 Sparse stochastic models: the step beyond Gaussianity	2
	1.3 From splines to stochastic processes, or when Schoenberg meets Lévy	5
	1.3.1 Splines and Legos revisited	5
	1.3.2 Higher-degree polynomial splines	8
	1.3.3 Random splines, innovations, and Lévy processes	9
	1.3.4 Wavelet analysis of Lévy processes and M -term approximations	12
	1.3.5 Lévy’s wavelet-based synthesis of Brownian motion	15
	1.4 Historical notes: Paul Lévy and his legacy	16
2	Roadmap to the book	19
	2.1 On the implications of the innovation model	20
	2.1.1 Linear combination of sampled values	20
	2.1.2 Wavelet analysis	21
	2.2 Organization of the book	22
3	Mathematical context and background	25
	3.1 Some classes of function spaces	25
	3.1.1 About the notation: mathematics vs. engineering	28
	3.1.2 Normed spaces	28
	3.1.3 Nuclear spaces	29
	3.2 Dual spaces and adjoint operators	32
	3.2.1 The dual of L_p spaces	33
	3.2.2 The duals of \mathcal{D} and \mathcal{S}	33
	3.2.3 Distinction between Hermitian and duality products	34
	3.3 Generalized functions	35
	3.3.1 Intuition and definition	35
	3.3.2 Operations on generalized functions	36
	3.3.3 The Fourier transform of generalized functions	37
	3.3.4 The kernel theorem	38

	3.3.5	Linear shift-invariant operators and convolutions	39
	3.3.6	Convolution operators on $L_p(\mathbb{R}^d)$	40
3.4		Probability theory	43
	3.4.1	Probability measures	43
	3.4.2	Joint probabilities and independence	44
	3.4.3	Characteristic functions in finite dimensions	45
	3.4.4	Characteristic functionals in infinite dimensions	46
3.5		Generalized random processes and fields	47
	3.5.1	Generalized random processes as collections of random variables	47
	3.5.2	Generalized random processes as random generalized functions	49
	3.5.3	Determination of statistics from the characteristic functional	49
	3.5.4	Operations on generalized stochastic processes	51
	3.5.5	Innovation processes	52
	3.5.6	Example: filtered white Gaussian noise	53
3.6		Bibliographical pointers and historical notes	54
4		Continuous-domain innovation models	57
	4.1	Introduction: from Gaussian to sparse probability distributions	58
	4.2	Lévy exponents and infinitely divisible distributions	59
	4.2.1	Canonical Lévy–Khintchine representation	60
	4.2.2	Deciphering the Lévy–Khintchine formula	64
	4.2.3	Gaussian vs. sparse categorization	68
	4.2.4	Proofs of Theorems 4.1 and 4.2	69
	4.3	Finite-dimensional innovation model	71
	4.4	White Lévy noises or innovations	73
	4.4.1	Specification of white noise in Schwartz’ space \mathcal{S}'	73
	4.4.2	Impulsive Poisson noise	76
	4.4.3	Properties of white noise	78
	4.5	Generalized stochastic processes and linear models	84
	4.5.1	Innovation models	84
	4.5.2	Existence and characterization of the solution	84
	4.6	Bibliographical notes	87
5		Operators and their inverses	89
	5.1	Introductory example: first-order differential equation	90
	5.2	Shift-invariant inverse operators	92
	5.3	Stable differential systems in 1-D	95
	5.3.1	First-order differential operators with stable inverses	96
	5.3.2	Higher-order differential operators with stable inverses	96
	5.4	Unstable N th-order differential systems	97
	5.4.1	First-order differential operators with unstable shift-invariant inverses	97
	5.4.2	Higher-order differential operators with unstable shift-invariant inverses	101
	5.4.3	Generalized boundary conditions	102

5.5	Fractional-order operators	104
5.5.1	Fractional derivatives in one dimension	104
5.5.2	Fractional Laplacians	107
5.5.3	L_p -stable inverses	108
5.6	Discrete convolution operators	109
5.7	Bibliographical notes	111
6	Splines and wavelets	113
6.1	From Legos to wavelets	113
6.2	Basic concepts and definitions	118
6.2.1	Spline-admissible operators	118
6.2.2	Splines and operators	120
6.2.3	Riesz bases	121
6.2.4	Admissible wavelets	124
6.3	First-order exponential B-splines and wavelets	124
6.3.1	B-spline construction	125
6.3.2	Interpolator in augmented-order spline space	126
6.3.3	Differential wavelets	126
6.4	Generalized B-spline basis	127
6.4.1	B-spline properties	128
6.4.2	B-spline factorization	136
6.4.3	Polynomial B-splines	137
6.4.4	Exponential B-splines	138
6.4.5	Fractional B-splines	139
6.4.6	Additional brands of univariate B-splines	141
6.4.7	Multidimensional B-splines	141
6.5	Generalized operator-like wavelets	142
6.5.1	Multiresolution analysis of $L_2(\mathbb{R}^d)$	142
6.5.2	Multiresolution B-splines and the two-scale relation	143
6.5.3	Construction of an operator-like wavelet basis	144
6.6	Bibliographical notes	147
7	Sparse stochastic processes	150
7.1	Introductory example: non-Gaussian AR(1) processes	150
7.2	General abstract characterization	152
7.3	Non-Gaussian stationary processes	158
7.3.1	Autocorrelation function and power spectrum	159
7.3.2	Generalized increment process	160
7.3.3	Generalized stationary Gaussian processes	161
7.3.4	CARMA processes	162
7.4	Lévy processes and their higher-order extensions	163
7.4.1	Lévy processes	163
7.4.2	Higher-order extensions of Lévy processes	166
7.4.3	Non-stationary Lévy correlations	167
7.4.4	Removal of long-range dependencies	169

	7.4.5	Examples of sparse processes	172
	7.4.6	Mixed processes	175
7.5		Self-similar processes	176
	7.5.1	Stable fractal processes	177
	7.5.2	Fractional Brownian motion through the looking-glass	180
	7.5.3	Scale-invariant Poisson processes	185
7.6		Bibliographical notes	187
8		Sparse representations	191
	8.1	Decoupling of Lévy processes: finite differences vs. wavelets	191
	8.2	Extended theoretical framework	194
	8.2.1	Discretization mechanism: sampling vs. projections	194
	8.2.2	Analysis of white noise with non-smooth functions	195
	8.3	Generalized increments for the decoupling of sample values	197
	8.3.1	First-order statistical characterization	199
	8.3.2	Higher-order statistical dependencies	200
	8.3.3	Generalized increments and stochastic difference equations	201
	8.3.4	Discrete whitening filter	202
	8.3.5	Robust localization	202
	8.4	Wavelet analysis	205
	8.4.1	Wavelet-domain statistics	206
	8.4.2	Higher-order wavelet dependencies and cumulants	208
	8.5	Optimal representation of Lévy and AR(1) processes	210
	8.5.1	Generalized increments and first-order linear prediction	211
	8.5.2	Vector-matrix formulation	212
	8.5.3	Transform-domain statistics	212
	8.5.4	Comparison of orthogonal transforms	216
	8.6	Bibliographical notes	222
9		Infinite divisibility and transform-domain statistics	223
	9.1	Composition of id laws, spectral mixing, and analysis of white noise	224
	9.2	Class C and unimodality	230
	9.3	Self-decomposable distributions	232
	9.4	Stable distributions	234
	9.5	Rate of decay	235
	9.6	Lévy exponents and cumulants	237
	9.7	Semigroup property	239
	9.7.1	Gaussian case	241
	9.7.2	S α S case	241
	9.7.3	Compound-Poisson case	241
	9.7.4	General iterated-convolution interpretation	241

	Contents	xi
9.8	Multiscale analysis	242
9.8.1	Scale evolution of the pdf	243
9.8.2	Scale evolution of the moments	244
9.8.3	Asymptotic convergence to a Gaussian/stable distribution	246
9.9	Notes and pointers to the literature	247
10	Recovery of sparse signals	248
10.1	Discretization of linear inverse problems	249
10.1.1	Shift-invariant reconstruction subspace	249
10.1.2	Finite-dimensional formulation	252
10.2	MAP estimation and regularization	255
10.2.1	Potential function	256
10.2.2	LMMSE/Gaussian solution	258
10.2.3	Proximal operators	259
10.2.4	MAP estimation	261
10.3	MAP reconstruction of biomedical images	263
10.3.1	Scale-invariant image model and common numerical setup	264
10.3.2	Deconvolution of fluorescence micrographs	265
10.3.3	Magnetic resonance imaging	269
10.3.4	X-ray tomography	272
10.3.5	Discussion	276
10.4	The quest for the minimum-error solution	277
10.4.1	MMSE estimators for first-order processes	278
10.4.2	Direct solution by belief propagation	279
10.4.3	MMSE vs. MAP denoising of Lévy processes	283
10.5	Bibliographical notes	286
11	Wavelet-domain methods	290
11.1	Discretization of inverse problems in a wavelet basis	291
11.1.1	Specification of wavelet-domain MAP estimator	292
11.1.2	Evolution of the potential function across scales	293
11.2	Wavelet-based methods for solving linear inverse problems	294
11.2.1	Preliminaries	295
11.2.2	Iterative shrinkage/thresholding algorithm	296
11.2.3	Fast iterative shrinkage/thresholding algorithm	297
11.2.4	Discussion of wavelet-based image reconstruction	298
11.3	Study of wavelet-domain shrinkage estimators	300
11.3.1	Pointwise MAP estimators for AWGN	301
11.3.2	Pointwise MMSE estimators for AWGN	301
11.3.3	Comparison of shrinkage functions: MAP vs. MMSE	303
11.3.4	Conclusion on simple wavelet-domain shrinkage estimators	312
11.4	Improved denoising by consistent cycle spinning	313
11.4.1	First-order wavelets: design and implementation	313
11.4.2	From wavelet bases to tight wavelet frames	315

xii	Contents	
	11.4.3 Iterative MAP denoising	318
	11.4.4 Iterative MMSE denoising	320
	11.5 Bibliographical notes	324
12	Conclusion	326
Appendix A	Singular integrals	328
	A.1 Regularization of singular integrals by analytic continuation	329
	A.2 Fourier transform of homogeneous distributions	331
	A.3 Hadamard’s finite part	332
	A.4 Some convolution integrals with singular kernels	334
Appendix B	Positive definiteness	336
	B.1 Positive definiteness and Bochner’s theorem	336
	B.2 Conditionally positive–definite functions	339
	B.3 Lévy–Khintchine formula from the point of view of generalized functions	342
Appendix C	Special functions	344
	C.1 Modified Bessel functions	344
	C.2 Gamma function	344
	C.3 Symmetric-alpha-stable distributions	346
	<i>References</i>	347
	<i>Index</i>	363

Preface

In the years since 2000, there has been a significant shift in paradigm in signal processing, statistics, and applied mathematics that revolves around the concept of sparsity and the search for “sparse” representations of signals. Early signs of this (r)evolution go back to the discovery of wavelets, which have now superseded classical Fourier techniques in a number of applications. The other manifestation of this trend is the emergence of data-processing schemes that minimize an ℓ_1 norm as opposed to the squared ℓ_2 norm associated with the traditional linear methods. A highly popular research topic that capitalizes on those ideas is compressed sensing. It is the quest for a statistical framework that would support this change of paradigm that led us to the writing of this book.

The cornerstone of our formulation is the classical innovation model, which is equivalent to the specification of stochastic processes as solutions of linear stochastic differential equations (SDE). The non-standard twist here is that we allow for non-Gaussian driving terms (white Lévy noise) which, as we shall see, has a dramatic effect on the type of signal being generated. A fundamental property, hinted in the title of the book, is that the non-Gaussian solutions of such SDEs admit a sparse representation in an adapted wavelet-like basis. While a sizable part of the present material is an outgrowth of our own research, it is founded on the work of Lévy (1930) and Gelfand (arguably, the second most famous Soviet mathematician after Kolmogorov), who derived general functional tools and results that are hardly known by practitioners but, as we argue in the book, are extremely relevant to the issue of sparsity. The other important source of inspiration is spline theory and the observation that splines and stochastic processes are ruled by the same differential equations. This is the reason why we opted for the innovation approach which facilitates the transposition of analytical techniques from one field to the other. While the formulation requires advanced mathematics that are carefully explained in the book, the underlying model has a strong engineering appeal since it constitutes the natural extension of the traditional filtered-white-noise interpretation of a Gaussian stationary process.

The book assumes that the reader has a good understanding of linear systems (ordinary differential equations, convolution), Hilbert spaces, generalized functions (i.e., inner products, Dirac impulses, linear operators), the Fourier transform, basic statistical signal processing, and (multivariate) statistics (probability density and characteristic functions). By contrast, there is no requirement for prior knowledge of splines, stochastic differential equations, or advanced functional analysis (function

spaces, Bochner’s theorem, operator theory, singular integrals) since these topics are treated in a self-contained fashion.

Several people have had a crucial role in the genesis of this book. The idea of defining sparse stochastic processes originated during the preparation of a talk for Martin Vetterli’s 50th birthday (which coincided with the anniversary of the launching of Sputnik) in an attempt to build a bridge between his signals with a finite rate of innovation and splines. We thank him for his long-time friendship and for convincing us to undertake this writing project. We are grateful to our former collaborator, Thierry Blu, for his precious help in the elucidation of the functional link between splines and stochastic processes. We are extremely thankful to Arash Amini, Julien Fageot, Pedram Pad, Qiyu Sun, and John-Paul Ward for many helpful discussions and their contributions to mathematical results. We are indebted to Emrah Bostan, Ulugbek Kamilov, Hagai Kirshner, Masih Nilchian, and Cédric Vonesch for turning the theory into practice and for running the signal- and image-processing experiments described in Chapters 10 and 11. We are most grateful to Philippe Thévenaz for his intelligent editorial advice and his spotting of multiple errors and inconsistencies, while we take full responsibility for the remaining ones. We also thank Phil Meyler, Sarah Marsh and Gaja Poggiogalli from Cambridge University Press, as well as John King for his careful copy-editing.

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The European Research Commission (ERC) and the Swiss National Science Foundation provided partial support throughout the writing of the book.

Notation

Abbreviations

ADMM	Alternating-direction method of multipliers
AL	Augmented Lagrangian
AR	Autoregressive
ARMA	Autoregressive moving average
AWGN	Additive white Gaussian noise
BIBO	Bounded input, bounded output
CAR	Continuous-time autoregressive
CARMA	Continuous-time autoregressive moving average
CCS	Consistent cycle spinning
DCT	Discrete cosine transform
fBm	Fractional Brownian motion
FBP	Filtered backprojection
FFT	Fast Fourier transform
FIR	Finite impulse response
FISTA	Fast iterative shrinkage/thresholding algorithm
ICA	Independent-component analysis
id	Infinitely divisible
i.i.d.	Independent identically distributed
IIR	Infinite impulse response
ISTA	Iterative shrinkage/thresholding algorithm
JPEG	Joint Photographic Experts Group
KLT	Karhunen–Loève transform
LMMSE	Linear minimum-mean-square error
LPC	Linear predictive coding
LSI	Linear shift-invariant
MAP	Maximum a posteriori
MMSE	Minimum-mean-square error
MRI	Magnetic resonance imaging
PCA	Principal-component analysis
pdf	Probability density function
PSF	Point-spread function
ROI	Region of interest

$S\alpha S$	Symmetric-alpha-stable
SDE	Stochastic differential equation
SNR	Signal-to-noise ratio
WSS	Wide-sense stationary

Sets

\mathbb{N}, \mathbb{Z}^+	Non-negative integers, including 0
\mathbb{Z}	Integers
\mathbb{R}	Real numbers
\mathbb{R}^+	Non-negative real numbers
\mathbb{C}	Complex numbers
\mathbb{R}^d	d -dimensional Euclidean space
\mathbb{Z}^d	d -dimensional integers

Various notation

j	Imaginary unit such that $j^2 = -1$
$\lceil x \rceil$	Ceiling: smallest integer at least as large as x
$\lfloor x \rfloor$	Floor: largest integer not exceeding x
$(x_1 : x_n)$	n -tuple (x_1, x_2, \dots, x_n)
$\ f\ $	Norm of the function f (see Section 3.1.2)
$\ f\ _{L_p}$	L_p -norm of the function f (in the sense of Lebesgue)
$\ a\ _{\ell_p}$	ℓ_p -norm of the sequence a
$\langle \varphi, s \rangle$	Scalar (or duality) product
$\langle f, g \rangle_{L_2}$	L_2 inner product
f^\vee	Reversed signal: $f^\vee(\mathbf{r}) = f(-\mathbf{r})$
$(f * g)(\mathbf{r})$	Continuous-domain convolution
$(a * b)[\mathbf{n}]$	Discrete-domain convolution
$\widehat{\varphi}(\omega)$	Fourier transform of φ : $\int_{\mathbb{R}^d} \varphi(\mathbf{r}) e^{-j\langle \omega, \mathbf{r} \rangle} d\mathbf{r}$
$\widehat{f} = \mathcal{F}\{f\}$	Fourier transform of f (classical or generalized)
$f = \mathcal{F}^{-1}\{\widehat{f}\}$	Inverse Fourier transform of \widehat{f}
$\overline{\mathcal{F}\{f\}}(\omega) = \mathcal{F}\{f\}(-\omega)$	Conjugate Fourier transform of f

Signals, functions, and kernels

$f, f(\cdot)$, or $f(\mathbf{r})$	Continuous-domain signal: function $\mathbb{R}^d \rightarrow \mathbb{R}$
φ	Generic test function in $\mathcal{S}(\mathbb{R}^d)$
$\psi_L = L^* \phi$	Operator-like wavelet with smoothing kernel ϕ
$s, \langle \varphi, s \rangle$	Generalized function $\mathcal{S}'(\mathbb{R}^d) \rightarrow \mathbb{R}$
μ_h	Measure associated with h : $\langle \varphi, h \rangle = \int_{\mathbb{R}^d} \varphi(\mathbf{r}) \mu_h(d\mathbf{r})$
δ	Dirac impulse: $\langle \varphi, \delta \rangle = \varphi(\mathbf{0})$
$\delta(\cdot - \mathbf{r}_0)$	Shifted Dirac impulse
β_L	Generalized B-spline associated with the operator L
φ_{int}	Spline interpolation kernel

$\beta_+^n = \beta_{D^{n+1}}$	Causal polynomial B-spline of degree n
$x_+^n = \max(0, x)^n$	One-sided power function
β_α	First-order exponential B-spline with pole $\alpha \in \mathbb{C}$
$\beta_{(\alpha_1:\alpha_N)}$	N th-order exponential B-spline: $\beta_{\alpha_1} * \cdots * \beta_{\alpha_N}$
$a, a[\cdot]$, or $a[\mathbf{n}]$	Discrete-domain signal: sequence $\mathbb{Z}^d \rightarrow \mathbb{R}$
$\delta[\mathbf{n}]$	Discrete Kronecker impulse

Spaces

\mathcal{X}, \mathcal{Y}	Generic vector spaces (normed or nuclear)
$L_2(\mathbb{R}^d)$	Finite-energy functions $\int_{\mathbb{R}^d} f(\mathbf{r}) ^2 \, d\mathbf{r} < \infty$
$L_p(\mathbb{R}^d)$	Functions such that $\int_{\mathbb{R}^d} f(\mathbf{r}) ^p \, d\mathbf{r} < \infty$
$L_{p,\alpha}(\mathbb{R}^d)$	Functions such that $\int_{\mathbb{R}^d} f(\mathbf{r}) (1 + \mathbf{r})^\alpha \, d\mathbf{r} < \infty$
$\mathcal{D}(\mathbb{R}^d)$	Smooth and compactly supported test functions
$\mathcal{D}'(\mathbb{R}^d)$	Distributions or generalized functions over \mathbb{R}^d
$\mathcal{S}(\mathbb{R}^d)$	Smooth and rapidly decreasing test functions
$\mathcal{S}'(\mathbb{R}^d)$	Tempered distributions (generalized functions)
$\mathcal{R}(\mathbb{R}^d)$	Bounded functions with rapid decay
$\ell_2(\mathbb{Z}^d)$	Finite-energy sequences $\sum_{\mathbf{k} \in \mathbb{Z}^d} a[\mathbf{k}] ^2 < \infty$
$\ell_p(\mathbb{Z}^d)$	Sequences such that $\sum_{\mathbf{k} \in \mathbb{Z}^d} a[\mathbf{k}] ^p < \infty$

Operators

Id	Identity
$D = \frac{d}{dt}$	Derivative
D_d	Finite difference (discrete derivative)
D^N	N th-order derivative
∂^n	Partial derivative of order $\mathbf{n} = (n_1, \dots, n_d)$
L	Whitening operator (LSI)
$\widehat{L}(\omega)$	Frequency response of L (Fourier multiplier)
ρ_L	Green's function of L
L^*	Adjoint of L such that $\langle \varphi_1, L\varphi_2 \rangle = \langle L^*\varphi_1, \varphi_2 \rangle$
L^{-1}	Right inverse of L such that $LL^{-1} = \text{Id}$
$h(\mathbf{r}_1, \mathbf{r}_2)$	Generalized impulse response of L^{-1}
L^{-1*}	Left inverse of L^* such that $(L^{-1*})L^* = \text{Id}$
L_d	Discrete counterpart of L
\mathcal{N}_L	Null space of L
P_α	First-order differential operator: $D - \alpha \text{Id}, \alpha \in \mathbb{C}$
$P_{(\alpha_1:\alpha_N)}$	Differential operator of order N : $P_{\alpha_1} \circ \cdots \circ P_{\alpha_N}$
Δ_α	First-order weighted difference
$\Delta_{(\alpha_1:\alpha_N)}$	N th-order weighted differences: $\Delta_{\alpha_1} \circ \cdots \circ \Delta_{\alpha_N}$
∂_τ^γ	Fractional derivative of order $\gamma \in \mathbb{R}^+$ and phase τ
$(-\Delta)^{\frac{\gamma}{2}}$	Fractional Laplacian of order $\gamma \in \mathbb{R}^+$
$I_p^{\gamma*}$	L_p -stable left inverse of $(-\Delta)^{\frac{\gamma}{2}}$

Probability

X, Y	Generic scalar random variables
\mathcal{P}_X	Probability measure on \mathbb{R} of X
$p_X(x)$	Probability density function (univariate)
$\Phi_X(x)$	Potential function: $-\log p_X(x)$
$\text{prox}_{\Phi_X}(x, \lambda)$	Proximal operator
$p_{\text{id}}(x)$	Infinitely divisible probability law
$\mathbb{E}\{\cdot\}$	Expected value operator
m_n	n th-order moment: $\mathbb{E}\{X^n\}$
κ_n	n th-order cumulant
$\widehat{p}_X(\omega)$	Characteristic function of X : $\mathbb{E}\{e^{j\omega X}\}$
$f(\omega)$	Lévy exponent: $\log \widehat{p}_{\text{id}}(\omega)$
$\nu(a)$	Lévy density
$p_{(X_1:X_N)}(\mathbf{x})$	Multivariate probability density function
$\widehat{p}_{(X_1:X_N)}(\boldsymbol{\omega})$	Multivariate characteristic function
$m_{\mathbf{n}}$	Moment with multi-index $\mathbf{n} = (n_1, \dots, n_N)$
$\kappa_{\mathbf{n}}$	Cumulant with multi-index \mathbf{n}
$H_{(X_1:X_N)}$	Differential entropy
$I(X_1, \dots, X_N)$	Mutual information
$D(p\ q)$	Kullback–Leibler divergence

Generalized stochastic processes

w	Continuous-domain white noise (innovation)
$\langle \varphi, w \rangle$	Generic scalar observation of innovation process
$f_{\varphi}(\omega)$	Modified Lévy exponent: $\log \widehat{p}_{\langle \varphi, w \rangle}(\omega)$
$\nu_{\varphi}(a)$	Modified Lévy density
s	Generalized stochastic process: $L^{-1}w$
u	Generalized increment process: $L_{\text{d}}s = \beta_L * w$
W	1-D Lévy process with $DW = w$
B_H	Fractional Brownian motion with Hurst index H
$\widehat{\mathcal{P}}_s(\varphi)$	Characteristic functional: $\mathbb{E}\{e^{j\langle \varphi, s \rangle}\}$
$\mathcal{B}_s(\varphi_1, \varphi_2)$	Correlation functional: $\mathbb{E}\{\langle \varphi_1, s \rangle \langle \varphi_2, s \rangle\}$
$R_s(\mathbf{r}_1, \mathbf{r}_2)$	Autocorrelation function: $\mathbb{E}\{s(\mathbf{r}_1)\overline{s(\mathbf{r}_2)}\}$