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Introduction and Main Concepts

This chapter is aimed at an audience that is not yet familiar with the area of nonlinear dynamics and its mathematical description. Its goal is to introduce the reader to the terminology and fundamental tenets adopted by the majority of scientists working in this area, as well as to provide theoretical and mathematical tools and background to better understand the subsequent parts of the book.

Concepts of nonlinear dynamics, such as dynamical systems, bifurcations, attractors, and Lyapunov exponents, will be briefly described. However, since the main topic of this book is synchronization, we present these auxiliary topics with the minimum of details and limit ourselves to a rather informal descriptive presentation. A reader interested in further deepening their general knowledge in nonlinear science and its applications is referred to the following books: Schuster and Just 2005; Baker and Gollub 1996; Ott 2002; Strogatz 2015; Fuchs 2013; Guckenheimer and Holmes 1983.

1.1 Dynamical Systems

From atoms to galaxies, at every length scale of study, one can distinguish relatively isolated self-organized structures referred to as *systems*. The world, both around and inside us, consists of many such systems. Most systems of interest are not fully isolated, but interact with each other. Their interactions may be due to fundamental physical forces, such as gravity or electromagnetism, collisions, or exchange of energy or matter.

In classical mechanics, the study of motion of bodies induced by external or internal forces is called *dynamics*, a word which originates from the Greek word $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$, meaning "power." In a more general context, we understand the term *dynamical* to be equivalent with *time-dependent*. Therefore, a *dynamical system* is a system that evolves over time. Hereby, we tacitly adopt the Newtonian concept of a globally defined time, i.e., we assume that the system variables are functions of time, treated as a universal parameter. In the context of mechanics, the system variables are typically positions and velocities of idealized mass points and, in the simplest case, the evolution of the system variables is determined by Newton's laws. However, in the following we will not restrict ourselves to mechanical systems, but will allow for more general system variables: for instance, concentrations of chemicals, intensity of a light beam, or temperature at a given point. Note that the system variables can be functions of both space and time.

Unless stated otherwise, we will focus on the case of *deterministic* systems, i.e., dynamical systems that are not influenced by noise. Mathematically, a dynamical system can be described by either differential or difference equations. In the former case, time flows continuously and the system is called a *continuous system* or *flow*. In the latter case, time changes discretely, and the system is known as a *discrete system*. In the following, we will describe the main features of dynamical systems by referring to popular examples.

1.1.1 Linear Dynamical Systems

Dynamical systems whose variables are linked by linear functions are called *linear systems*. The temporal evolution of a continuous linear system characterized by n system variables is generated by a set of n ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{aligned} \tag{1.1}$$

where $x_i = x_i(t)$ are the time-dependent system variables, $\dot{x}_i \equiv \mathrm{d}x_i/\mathrm{dt}$ are their time derivatives, and a_{ij} are constant coefficients.

In vector form, Equation 1.1 can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-dimensional vector, and $\mathbf{A}$ is a constant matrix.

When the system dynamics are defined in terms of discrete times, i.e., when the current state of the system is iteratively determined by the previous one, the dynamics are instead described by difference equations, or *iterative maps*. The variables exhibit a mapping form, as $\mathbf{x}$ varies in discrete steps:

$$\mathbf{x_{i+1}} = \mathbf{A}\mathbf{x_i}. \tag{1.3}$$

Linear systems can be solved exactly. The solution of Equation 1.2 has an exponential form and can be found using the set of eigenvalues λ, given by the determinant equation

$$\det(A - \lambda\mathbf{I}) = 0, \tag{1.4}$$

where $\mathbf{I}$ is the identity matrix, and the eigenvectors $\mathbf{v_i}$ satisfy the equation

$$\mathbf{A}\mathbf{v_i} = \lambda_i \mathbf{v_i}\,. \tag{1.5}$$

The eigenvalues λ represent powers of the exponential components of the solution, and the eigenvectors are their coefficients.

Pure linear systems, however, do not exist in nature. Like a point mass, they are just mathematical approximations. The dynamics of linear systems, indeed, are not rich enough to describe the most commonly observed behaviors, such as periodic oscillations, bifurcations, synchronization, and chaos. The asymptotic state of a bounded linear system, reached for $t \to \infty$, is only a steady state, i.e., a fixed equilibrium point that can be either stable or unstable. The stability properties of dynamical systems will be described in Section 1.4.

1.1.2 Nonlinear Dynamical Systems

If a system is characterized by variables that depend nonlinearly on each other, the motion can become very complex. Such systems are called *nonlinear dynamical systems*. Mathematically, a nonlinear continuous dynamical system is described by

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1, x_2, \ldots, x_n), \\
\dot{x}_2 &= F_2(x_1, x_2, \ldots, x_n), \\
&\vdots \\
\dot{x}_n &= F_n(x_1, x_2, \ldots, x_n),
\end{aligned} \tag{1.6}$$

where F_i are functions that couple the variables among them. If at least one of these functions is nonlinear, the system in Equation 1.6 is said to be nonlinear.

In vector form, a nonlinear dynamical system can be described as

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)), \tag{1.7}$$

where $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is a vector function: $\mathbb{R}^n \to \mathbb{R}^n$.

The time evolution of the system describes a *trajectory*, or *orbit*, in the Euclidean space of the n variables $\mathbf{x} \in \mathbb{R}^n$, or *phase space*. Each point in the phase space represents a unique state of the system. In the case of three-dimensional systems, one can visualize directly the trajectory in three coordinates (x_1, x_2, x_3), while for systems with $n > 3$, visualization of the orbit is only possible by means of projections of the phase space on planes (or hyperplanes) of two or three of the system's variables.

Since at any given time the system state is described by a vector defined by the functions and parameters in Equation 1.7, the system evolution is *deterministic*.

In other words, for any fixed initial condition $\mathbf{x}_0 \in \mathbb{R}^n$, the system always follows a unique path, which therefore can never intersect the paths originating from different initial conditions. However, it has to be remarked that a strictly deterministic system is only a theoretical idealization, because random components or fluctuations are always present in nature, and are sometimes accounted for via perturbation theory. Furthermore, in practice, exact knowledge about the future state of a system is restricted by the precision with which the initial state can be measured, especially for chaotic systems characterized by a strong dependence on initial conditions.

1.1.3 Autonomous and Nonautonomous Systems

A dynamical system that contains a time-dependent function is called *nonautonomous*; otherwise, the system is said to be *autonomous*. Every nonautonomous system can be transformed into an autonomous system by adding an additional degree of freedom proportional to the time. As an example, let us consider such a transformation when applied to a CO_2 laser model. Under loss modulation, this laser represents a nonautonomous (or driven) system described as (Chizhevsky et al. 1997; Pisarchik and Corbalán 1999)

$$\begin{aligned} \dot{x} &= \tau^{-1} x \left(y - k_0 - k_m \sin(2\pi f_m t) \right) , \\ \dot{y} &= (y_0 - y)\gamma - yx \, . \end{aligned} \tag{1.8}$$

In these equations, x is proportional to the radiation density, y and y_0 stand for the gain and the unsaturated gain in the active medium, τ is the light half round-trip time in the laser cavity, γ is the gain decay rate, k_0 is the constant portion of the losses, and k_m and f_m are the modulation amplitude and frequency. The system is nonlinear because of the yx coupling term.

Introducing the additional variable $z = 2\pi f_m t$, one can convert the two-dimensional nonautonomous system in Equation 1.8 into the three-dimensional autonomous system:

$$\begin{aligned} \dot{x} &= \tau^{-1} x (y - k_0 - k_m \sin z), \\ \dot{y} &= (y_0 - y)\gamma - yx, \\ \dot{z} &= 2\pi f_m . \end{aligned} \tag{1.9}$$

It is clear that, by generalization of this procedure, any nth-order nonautonomous system can be transformed into an $(n + 1)$-dimensional autonomous system.

1.1.4 Conservative and Dissipative Systems

A dynamical system is said to be conservative (dissipative) if a unitary volume of initial conditions produces orbits whose images in time are contained within a constant (contracting) volume of the phase space. Although all real dynamical systems are dissipative, quantum mechanics mostly deals with conservative, or Hamiltonian, systems. The most notable examples of conservative systems are undamped pendula, sets of point masses interacting under Newton's gravitational force, and nonrelativistic charged particles in an electromagnetic field.

Dissipation arises from any kind of loss, quite often due to internal friction. In dissipative dynamical systems, the potential, or energy, goes from an initial form to a final asymptotic one. After a sufficiently long period – known as *transient* time – has elapsed, the trajectory of the dissipative system is found in a subset of the phase space that is said to be the system's *attractor*. As the energy in conservative systems is preserved, they do not have attractors.

A typical example of a dissipative linear system is a damped harmonic oscillator, described by

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{1.10}$$

where m is the mass, c is the viscous damping coefficient, and k is the elastic constant. These coefficients define the undamped angular frequency

$$\omega_0 = \sqrt{k/m} \tag{1.11}$$

and the damping ratio

$$\zeta = \frac{c}{2\sqrt{mk}}. \tag{1.12}$$

When the new variables $x_1 = x$ and $x_2 = \dot{x}$ are introduced, Equation 1.10 generates a system of two first-order differential equations:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -(c/m)x_2 - \omega_0^2 x_1. \end{aligned} \tag{1.13}$$

The damping ratio ζ determines the transient behavior of the oscillator. If $\zeta = 0$, the oscillator is undamped, the system is conservative, and the solutions are oscillations that continue indefinitely with frequency ω_0. If $0 < \zeta < 1$, the oscillator is *underdamped*, and its oscillations have a frequency $\omega = \omega_0\sqrt{1-\zeta^2}$. Otherwise, the system returns to its equilibrium without oscillating, and the oscillator is known as either *overdamped* (if $\zeta > 1$) or *critically damped* (if $\zeta = 1$).

All systems considered in this book behave like undamped or underdamped oscillators, since speaking of synchronization for fixed points is meaningless.

1.2 Chaotic Systems

Nonlinear differential equations are very difficult (or even impossible) to solve analytically, and until computer simulations became possible, chaotic solutions could not be calculated (Lorenz 1963).

By chaotic solutions, we generally denote those trajectories that have a critical dependence on the initial conditions. This means that if one considers any two trajectories originating from two nearby initial conditions (whose Euclidean distance in phase space is arbitrarily small), these trajectories *exponentially* separate in time, i.e., the distance between the two actual states of the system grows exponentially over time.

To discuss the main concepts of chaotic dynamics, let us consider a three-dimensional ($n = 3$) nonlinear Rössler oscillator, described by the system (Rössler 1977)

$$\begin{aligned} \dot{x} &= -\omega y - z, \\ \dot{y} &= \omega x + a y, \\ \dot{z} &= b + z(x - c). \end{aligned} \tag{1.14}$$

This oscillator is often used for studying synchronization because its natural frequency ω is directly included in the equations as a parameter.

In spite of its apparent simplicity, the system in Equation 1.14 exhibits very rich dynamics. For a large set of parameters, such as, for example, $a = 0.16$, $b = 0.1$, $c = 8.5$, and $\omega = 1$, the motion of the system Equation 1.14 is chaotic.

The dynamics of a nonlinear system can be visualized and characterized by the following tools: (i) time series, (ii) phase-space portrait, and (iii) power spectrum. Let us consider each of these tools separately.

1.2.1 Time Series

Time series describe the temporal evolution of a system variable. Figure 1.1 shows the time series of all three variables of Equation 1.14 in the chaotic state given by the parameter values defined above. Although the oscillations of each variable are irregular, they are correlated due to their functional dependence in Equation 1.14, even if we cannot see it at a first glance.

Time series have proven to be a good tool for synchronization analysis, to extract meaningful statistics and other data characteristics. Time series analysis can be carried out either in the time domain or in the frequency domain. The former includes cross-correlation analyses that are frequently used to quantify synchronization of coupled systems (see Section 2.3). The latter includes spectral analysis, which is described below. Time series analysis can also be used to reconstruct attractors from experimental data.

1.2.2 Phase Space

The concept of phase space was developed in the nineteenth century thanks to the contributions of Ludwig Boltzmann, Henri Poincaré, James Maxwell, and Josiah Gibbs to statistical mechanics and Hamiltonian mechanics. In nonlinear dynamics,

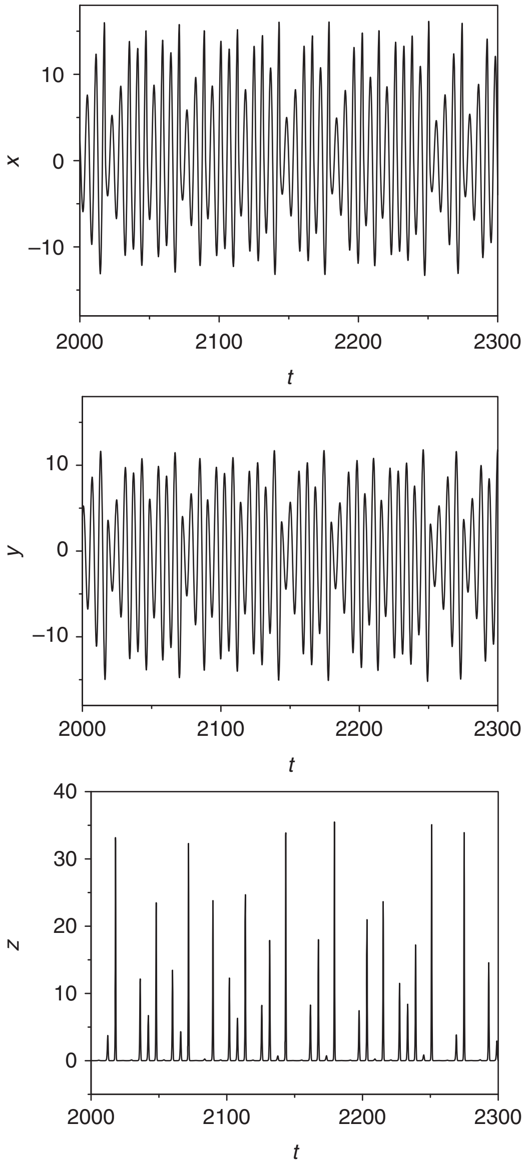

Figure 1.1 Time series of three variables of the Rössler oscillator, Equation 1.14, in the chaotic regime for $a = 0.16$, $b = 0.1$, $c = 8.5$, and $\omega = 1$.

the phase space is a space whose coordinates correspond to the system variables. The system trajectory in the phase space represents all possible states during an infinite time evolution. The phase space dimension is equal to the number of system variables. The phase space trajectory of the chaotic Rössler oscillator is shown in Figure 1.2.

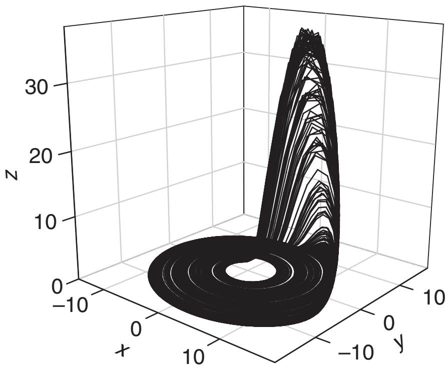

Figure 1.2 The chaotic trajectory of the Rössler oscillator for $a = 0.16$, $b = 0.1$, $c = 8.5$, and $\omega = 1$, contained within a certain region of the phase space (x, y, z).

When the phase space has a very large dimension, the trajectory of a system is often visualized by projections onto the subspace corresponding to two or three variables. These projections are called *phase portraits*.

1.2.3 Power Spectrum

Another way to visualize the dynamics of a system is via a *power* or *frequency spectrum* of one of the system variables. The power spectrum can be obtained by means of the *Fourier transform* of the time series. The Fourier transform $\mathcal{F}$, named after Joseph Fourier (1768–1830), is a mathematical transformation employed to transform a signal from a time domain to a frequency domain. A reverse operation $\mathcal{F}^{-1}$ is also possible. Mathematically, the direct and inverse Fourier transforms are defined as

$$\mathcal{F}\left(x\left(t\right)\right) \equiv X\left(\omega\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\left(t\right) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t\,, \tag{1.15}$$

$$\mathcal{F}^{-1}\left(X\left(\omega\right)\right) \equiv x\left(t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X\left(\omega\right) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega\,. \tag{1.16}$$

These integrals exist if three conditions are met, namely:

(i) $x\left(t\right)$ is piecewise continuous;
(ii) $x\left(t\right)$ is piecewise differentiable;
(iii) $x\left(t\right)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} \left|x\left(t\right)\right| \mathrm{d}t$ is finite.

Then, the power spectrum is defined as

$$S\left(\omega\right) = X^{*}\left(\omega\right) X\left(\omega\right) = \left|X\left(\omega\right)\right|^{2}\,, \tag{1.17}$$

where X^{*} is the complex conjugate of X.

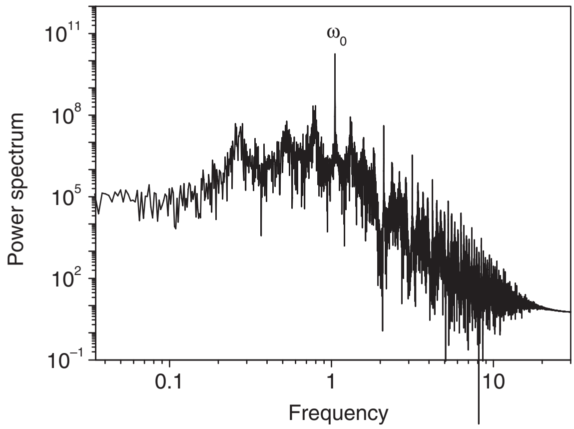

Figure 1.3 Power spectrum of the y variable of a Rössler oscillator with the parameter choice $a = 0.16$, $b = 0.1$, $c = 8.5$, and $\omega = 1$. Notice that the spectrum of the chaotic signal is continuous, and has a peak at the dominant frequency ω_0.

Figure 1.3 shows the power spectrum of the y variable of a Rössler oscillator with the usual parameters, obtained from the time series of Figure 1.1. The power spectrum of the chaotic regime has a maximum at a dominant frequency ω_0. Due to nonlinearity, ω_0 differs from ω by a small amount.

1.3 Attractors

Differential and difference nonlinear equations that describe dynamical systems give rise to many types of solutions, both stable and unstable. Stable solutions encountered in nonlinear dynamics are *attractors*: asymptotically stable volumes of the phase space toward which a system evolves, when starting from a set of initial conditions known as the attractor's *basin of attraction*.

1.3.1 Types of Attractors

Attractors are therefore portions (or subsets) of the phase space. The simplest possible attractor is a stable fixed point, which can be found even in linear systems (Section 1.1). Nonlinearity, however, allows for more complex and interesting attractors. They come in different geometric shapes in phase space, such as limit cycles (periodic orbits), toroids, and miscellaneous manifolds, and may even have a fractal structure (strange attractors).

A single nonlinear dynamical system can exhibit different attractors, depending on the values chosen for its parameters. Let us illustrate the case with the Rössler oscillator of Equation 1.14. In Figure 1.4 we show three different attractors in the phase space. Each of these attractors is determined by the parameter c, while keeping the other parameters unchanged.

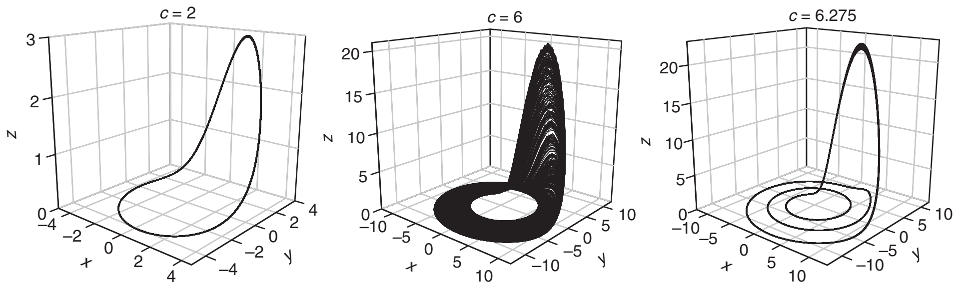

Figure 1.4 Types of attractors of a Rössler oscillator for different values of the parameter c. (a) A period-one orbit is generated for $c = 2$; (b) a chaotic attractor is generated for $c = 6$; and (c) a period-three orbit is generated for $c = 6.275$. The other parameters are $a = 0.16$, $b = 0.1$, and $\omega = 1$.

Some nonlinear systems even allow coexistence of attractors. While all parameters remain constant, the attractor changes according to the initial conditions. Such systems are called *multistable*. Multistability can be revealed not only by changing initial conditions, but also by varying a system parameter back and forth (continuation method; see Seydel 1988), or by adding noise that converts the multistable system to a metastable one (noise-induced multistate intermittency; see Pisarchik et al. 2012a).

1.3.2 Basins of Attraction and Poincaré Maps

The basin of attraction of an attractor is the set of initial conditions that lead the asymptotic trajectory of the system to the attracted state. For our example of Equation 1.14, the basin of attraction of the chaotic attractor has a very sophisticated structure in the three-dimensional phase space.

A visualization of the attractor can be given in a two-dimensional space by plotting the intersection points of trajectories with a plane corresponding to a certain (fixed) value of one of the system variables (say, z). Such a plane is called the *Poincaré section*, named after French mathematician Henri Poincaré (1854–1912). The attractor can be visualized on the Poincaré section by plotting the value of the function each time it passes through it in a specific direction. This map is called the *Poincaré map*, and it is a lower-dimensional subspace transversal to the system flow in phase space.

The Poincaré section is a very useful tool for the analysis of nonlinear dynamics, as well as for revealing the attractor's structure in spaces where dimensions are reduced by one. Indeed, given a flow, a Poincaré map can always be constructed, composed by intersection points $\mathbf{x_j}(t_j)$ on the Poincaré section in one direction at discrete times t_j. This procedure converts the flow Equation 1.7 to the map $\mathbf{x_{j+1}}(t_{j+1}) = \mathbf{P}(\mathbf{x}_j(t_j))$.