1

Wave fundamentals

This chapter covers the fundamental concepts of waves. As with all the chapters in the book, you can read the sections within this chapter in any order, or you can skip them entirely if you're already comfortable with this material. But if you're working through one of the later chapters and you find that you're uncertain about some aspect of the discussion, you can turn back to the relevant section of this chapter.

In the first two sections of this chapter you'll be able to review the basic definitions and terminology of waves (Section 1.1) and the relationships between wave parameters (Section 1.2). Later sections cover topics that serve as the foundation on which you can build your understanding of waves, including vectors (Section 1.3), complex numbers (Section 1.4), the Euler relations (Section 1.5), wavefunctions (Section 1.6), and phasors (Section 1.7).

1.1 Definitions

When you're embarking on a study of new topic, it's always a good idea to make sure you understand the terminology used by people who discuss that topic. Since this book is all about waves, a reasonable place to start is by asking the question "What exactly is a wave?"

Here are some of the answers to that question that you may encounter in the literature.

"A classical traveling wave is a self-sustaining disturbance of a medium, which moves through space transporting energy and momentum." [6].

"What is required for a physical situation to be referred to as a wave is that its mathematical representation give rise to a partial differential equation of a particular form, known as *the wave equation*." [9].

"[The essential feature of wave motion is that a] condition of some kind is transmitted from one place to another by means of a medium, but the medium itself is not transported." [4].

"[A wave is] each of those rhythmic alternations of disturbance and recovery of configuration."[1]

Although there's not a great deal of commonality in these definitions of a wave, each contains an element that can be very helpful when you're trying to decide whether some phenomenon can (or should) be called a wave.

The most common defining characteristic is that a wave is a *disturbance* of some kind, that is, a change from the equilibrium (undisturbed) condition. A string wave disturbs the position of segments of the string, a sound wave disturbs the ambient pressure, an electromagnetic wave disturbs the strengths of the electric and magnetic fields, and matter waves disturb the probability that a particle exists in the vicinity.

In *propagating* or *traveling* waves, the wave disturbance must move from place to place, carrying energy with it. But you should be aware that combinations of propagating waves can produce non-propagating disturbances, such as those of a standing wave (you can read more about this in Section 3.2 of Chapter 3).

In *periodic* waves, the wave disturbance repeats itself in time and space. So, if you stay in one location and wait long enough, you're sure to see the same disturbance as you've seen previously. And if you take an instantaneous snapshot of the wave, you'll be able to find different locations with the same disturbance. But combinations of periodic waves can add up to non-periodic disturbances such as a wave pulse (which you can read about in Section 3.3 of Chapter 3).

Finally, in *harmonic* waves, the shape of the wave is sinusoidal, meaning that it takes the form of a sine or cosine function. You can see plots of a sinusoidal wave in space and time in Fig. 1.1.

So waves are disturbances that may or may not be propagating, periodic, and harmonic. But whatever the type of wave, there are a few basic parameters that you should make sure you understand. Here's a little FAQ that you may find helpful.

Q: How far is it from one crest to the next?

A: λ (Greek letter "lambda"), the **wavelength**. Wavelength is the amount of distance per cycle and has dimensions of length; in SI,[2] the units

[1] Oxford English Dictionary.

[2] "SI" stands for "Système International d'unités", the standard metric reference system of units.

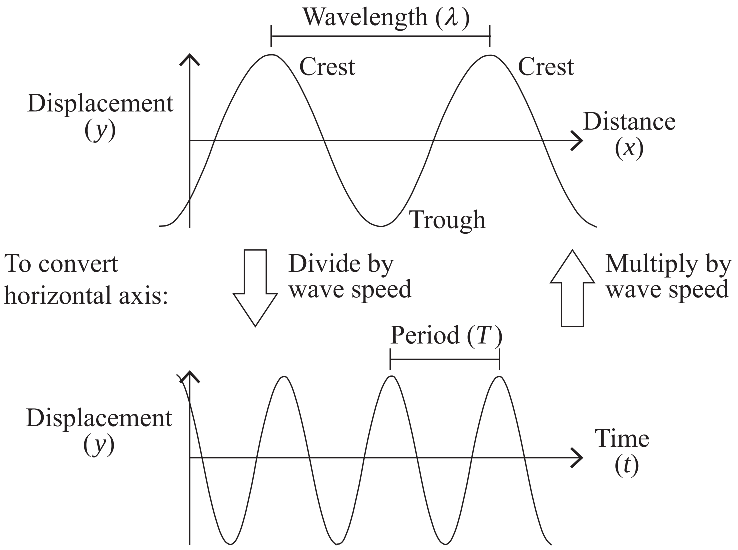

Figure 1.1 An example of a sinusoidal wave plotted in space and time.

of length are meters (m). Shouldn't this technically be "meters/cycle"? Yes, but since people know you're taking about waves when you mention wavelength, the "per cycle" is usually assumed and not explicitly kept in the units.

Q: How long in time is it between crests?

A: T (sometimes you'll see this written as P), the **period**. Period is the amount of time per cycle and has units of time, seconds (s) in SI. Again, this is really "seconds per cycle", but the "per cycle" is assumed and usually dropped.

Q: How often do crests come by?

A: f, the **frequency**. If you count how many wave crests pass by a given place in a certain amount of time, you are measuring f. Thus frequency is the number of cycles per amount of time and has units of one over time (technically cycles per unit time, but again, "cycles" is assumed and may be omitted). So in SI you'll see the units of frequency either as cycles/sec or 1/s, which are also called hertz (Hz). The frequency of a wave is the inverse of the wave's period (T).

An illustration of the meaning of wavelength, wave period, and frequency (and how they're measured) is shown in Fig. 1.2.

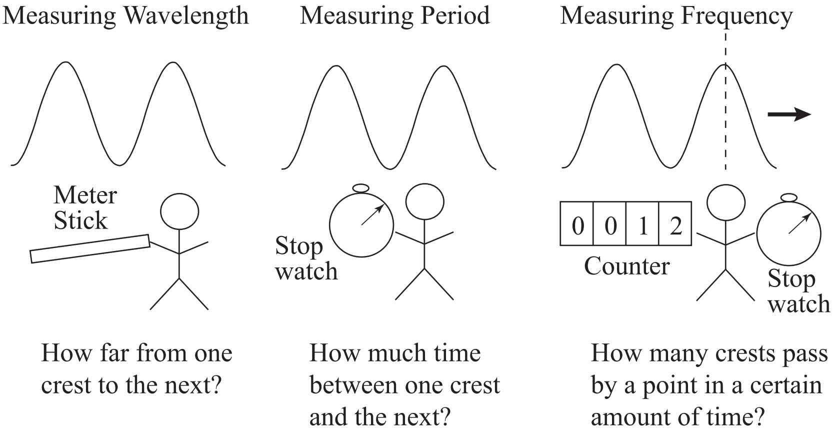

Figure 1.2 Measuring wave parameters.

Q: How big is the wave at any given place or time?

A: y, the **displacement**. Displacement is the amount of disturbance from equilibrium produced by the wave; its value depends on the place and time at which you measure the wave (and so is a function of x and t for a wave moving along the x-axis). The units of displacement depend on exactly what kind of a wave it is: waves on strings have displacements with units of distance (see Chapter 4), electromagnetic waves have displacements with units of electric and magnetic field strength (see Chapter 5), and one-dimensional quantum-mechanical matter waves have displacement with units of one over the square root of length (see Chapter 6).

Q: What is the biggest the wave ever gets?

A: A, the **amplitude**. Amplitude is a special value related to the displacement that occurs at the peak of a wave. We say "related to" because there are several different types of amplitude. "Peak" amplitude is the maximum displacement from equilibrium; this is measured from the equilibrium value to the top of the highest peak or the bottom of the deepest trough. "Peak-to-peak" amplitude is the difference between a positive peak and a negative peak, measured from crest to trough. And "rms" amplitude is the root-mean-square value of the displacement over one cycle. For sinusoidal waves, the peak-to-peak amplitude is twice as big as the peak amplitude, and the rms amplitude is 0.707 times the peak amplitude. Amplitude has the same units as displacement.

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Q: How fast is the wave moving?

A: v, the **wave speed**. Usually, when authors refer to wave speed, they're talking about **phase speed**: How fast does a given point on a wave move? For example, if you measure how long it takes for one crest of a wave to travel a certain distance, you're measuring the phase speed of the wave. A different speed, **group speed**, is important for groups of waves called wave packets whose shape may change over time; you can read more about this in Section 3.4 of Chapter 3.

Q: What determines which part of a wave is at a given place at a certain time?

A: ϕ (Greek letter "phi"), the **phase**. If you specify a place and time, the phase of the wave tells you whether a crest, a trough, or something in between will appear at that place and time. In other words, phase is the argument of the function that describes the wave (such as $\sin\phi$ or $\cos\phi$). Phase has SI units of radians and values between 0 and $\pm 2\pi$ over one cycle (you may also see phase expressed in units of degrees, in which case one cycle $= 360° = 2\pi$ radians).

Q: What determines the starting point of a wave?

A: ϵ (Greek letter "epsilon"), or ϕ_0 ("phi-zero"), the **phase constant**. At the time $t = 0$ and location $x = 0$, the phase constant ϵ or ϕ_0 tells you the phase of the wave. If you encounter two waves that have the same wavelength, frequency, and speed but are "offset" from one another (that is, they don't reach a peak at the same place or time), those waves have different phase constants. A cosine wave, for example, is just a sine wave with a phase-constant difference of $\pi/2$, or 90°.

Q: All this sounds suspiciously like phase is related to some kind of angle.

A: That's not a question, but you're right, which is why phase is sometimes called "phase angle". The next two definitions should help you understand that.

Q: What relates a wave's frequency or period to angles?

A: ω (Greek letter "omega"), the **angular frequency**. The angular frequency tells you how much angle the phase of the wave advances in a given amount of time, so the SI units of angular frequency are radians per second. Angular frequency is related to frequency by the equation $\omega = 2\pi f$.

Q: What relates a wave's wavelength to angles?

A: k, the **wavenumber**. The wavenumber tells you how much the phase of the wave advances in a given amount of distance, so wavenumber has SI units of radians per meter. Wavenumber is related to wavelength by the equation $k = 2\pi/\lambda$.

1.2 Basic relationships

Many of the basic wave parameters defined in the previous section are related to one another through simple algebraic equations. For example, the frequency (f) and the period (T) are related by

$$f = \frac{1}{T}. \tag{1.1}$$

This equation tells you that frequency and period are *inversely* proportional. This means that longer period corresponds to lower frequency, and shorter period corresponds to higher frequency.

You can verify that Eq. (1.1) is dimensionally consistent by recalling from Section 1.1 that the units of frequency are cycles/second (often expressed simply as 1/s) and the units of period are just the inverse: seconds/cycle (usually expressed as "s"). So the dimensions of Eq. (1.1) in SI units are

$$\left[\frac{\text{cycles}}{\text{seconds}}\right] = \left[\frac{1}{\text{seconds/cycle}}\right].$$

Another simple but powerful equation relates the wavelength (λ) and frequency (f) of a wave to the wave's speed (v). That equation is

$$\lambda f = v. \tag{1.2}$$

The basis for this equation can be understood by considering the fact that speed equals distance divided by time, and a wave covers a distance of one wavelength in a time interval of one period. Hence $v = \lambda/T$, and since $T = 1/f$, this is the same as $v = \lambda f$. It also makes physical sense, as you can see by considering a wave that has long wavelength and high frequency. In that case, the speed of the wave must be high, for how else could those far-apart crests (long wavelength) be coming past very often (high frequency)? Now think about a wave for which the wavelength and frequency are both small. Since those closely spaced crests (short wavelength) are not coming past very often (low frequency), the wave must be moving slowly.

To see that the dimensions are balanced in Eq. (1.2), consider the units of wavelength multiplied by the units of frequency:

$$\left[\frac{\text{meters}}{\text{cycle}}\right]\left[\frac{\text{cycles}}{\text{second}}\right] = \left[\frac{\text{meters}}{\text{second}}\right],$$

which are the units of speed.

So Eq. (1.2) allows you to find the speed of a wave if you know the wave's wavelength and frequency. But, as you study waves, you're likely to encounter many situations in which you're dealing with waves of the same type that

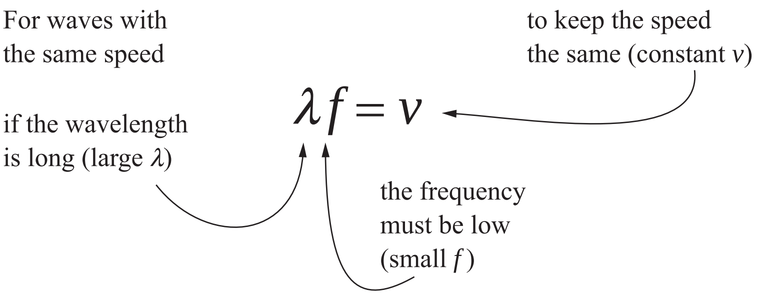

Figure 1.3 The relationship of wavelength to frequency for waves of the same speed.

are moving at the same speed (such as electromagnetic waves in a vacuum, which all travel at the speed of light). In such cases, the waves may have different wavelength (λ) and frequency (f), but the *product* of the wavelength and frequency must equal the wave speed.

This means that as long as the wave speed (v) is constant, waves with longer wavelength (large λ) must have lower frequency (small f). Likewise, for waves with the same speed, if the wavelength is short (small λ), the frequency must be high (large f). This concept is so important we've written it as an "expanded equation" in Fig. 1.3.

For sound waves (which have constant speed under certain circumstances), frequency corresponds to pitch. So low-pitch sounds (such as the bass notes of a tuba or the rumble of a passing truck) must have long wavelength, and high-pitch sounds (such as the tweets of a piccolo or Mickey Mouse's voice) must have short wavelength.

For electromagnetic waves in the visible portion of the spectrum, frequency corresponds to color. So the relationship between wavelength, frequency, and speed means that low-frequency (red) light has longer wavelength than high-frequency (blue) light.

There are two additional equations that are very useful when you're working on wave problems. The first of these is the relationship between frequency (f), period (T), and angular frequency (ω):

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{1.3}$$

You can see from this equation that angular frequency has dimensions of angle over time (SI units of rad/s), consistent with the definition of this parameter in Section 1.1. So frequency (f) tells you the number of cycles per second, and angular frequency (ω) tells you the number of radians per second.

Here's why the angular frequency (ω) of a wave is a useful parameter. Let's say you want to know how much the phase of a wave will change at a certain location in a given amount of time (Δt). To find that phase change ($\Delta\phi$), just multiply the angular frequency (ω) by the time interval (Δt):

$$(\Delta\phi)_{\text{constant } x} = \omega\,\Delta t = \left(\frac{2\pi}{T}\right)\Delta t = 2\pi\left(\frac{\Delta t}{T}\right), \tag{1.4}$$

where the subscript "constant x" is a reminder that this change in phase is due only to advancing time. If you change location, there will be an additional phase change as described below, but for now we're considering the phase change at one location (constant x).

At this point, it may help you to step back from Eq. (1.4) and take a look at the $\Delta t/T$ term. This ratio is just the fraction of a full period (T) that the time interval Δt represents. Since the phase change during a full period is 2π radians, multiplying this fraction ($\Delta t/T$) by 2π radians gives you the number of radians that the wave phase has advanced during the time interval Δt.

Example 1.1 *How much does the phase of a wave with period (T) of 20 seconds change in 5 seconds?*

Since the wave period T is 20 seconds, a time interval Δt of 5 seconds represents 1/4 period ($\Delta t/T = 5/20 = 1/4$). Multiplying this fraction by 2π gives $\pi/2$ radians. Thus the phase of the wave advances by $\pi/2$ radians (90°) every 5 seconds.

This illustrates why angular frequency (ω) can be thought of as a "time-to-phase converter". Given any amount of time t, you can convert that time to phase change by finding the product ωt.

The final important relationship of this section concerns wavenumber (k) and wavelength (λ). The relationship between these parameters is

$$k = \frac{2\pi}{\lambda}. \tag{1.5}$$

This equation shows that wavenumber has the dimensions of angle over distance (with SI units of rad/m). It also suggests that wavenumber can be used to convert distance to phase change, just as angular frequency can be used to convert time to phase change.

To find the phase change $\Delta\phi$ over a given distance at a certain time, multiply the wavenumber k by a distance interval Δx:

$$(\Delta\phi)_{\text{constant } t} = k\,\Delta x = \left(\frac{2\pi}{\lambda}\right)\Delta x = 2\pi\left(\frac{\Delta x}{\lambda}\right), \tag{1.6}$$

where the subscript "constant t" is a reminder that this change in phase is due only to changing location (as described above, there will be an additional phase change due to the passage of time).

Just as the term $\Delta t/T$ gives the fraction of a full cycle represented by the time interval Δt, the term $\Delta x/\lambda$ gives the fraction of a full cycle represented by the distance interval Δx. Thus the wavenumber k serves as a "distance-to-phase converter", allowing you to convert any distance x to a phase change by forming the product kx.

With an understanding of the meaning of the wave parameters and relationships described in this and the previous section, you're almost ready for a discussion of wavefunctions. But that discussion will be more meaningful to you if you also have a basic understanding of vector concepts, complex numbers, and the Euler relations. Those are the subjects of the next three sections.

1.3 Vector concepts

Before getting into complex numbers and Euler's relation, we think a discussion of basic vector concepts will provide a helpful foundation for those topics. That's because every complex number can be considered to be the result of vector addition, which is described later in this section. Furthermore, some waves involve vector quantities (such as electric and magnetic fields), and a quick review of the basics of vectors may help you understand those waves.

So what exactly is a vector? For many physics applications, you can think of a vector simply as a quantity that includes both a magnitude (how much) and a direction (which way). For example, speed is not a vector quantity; it's called a "scalar" quantity because it has magnitude (how fast an object is moving) but no direction. But velocity is a vector quantity, because velocity includes both speed and direction (how fast an object is moving and in which direction).

The are many other quantities that can be represented by vectors, including acceleration, force, linear momentum, angular momentum, electric fields, and magnetic fields. Vector quantities are often represented pictorially as arrows, in which the length of the arrow is proportional to the magnitude of the vector and the orientation of the arrow shows the direction of the vector. In text, vector quantities are usually indicated either using bold script (such as **A**) or by putting an arrow over the variable name (such as $\vec{A}$).

Just as you can perform mathematical operations such as addition, subtraction, and multiplication with scalars, you can also do these operations with vectors. The two operations most relevant to using vectors to understand complex numbers are vector addition and multiplication of a vector by a scalar.

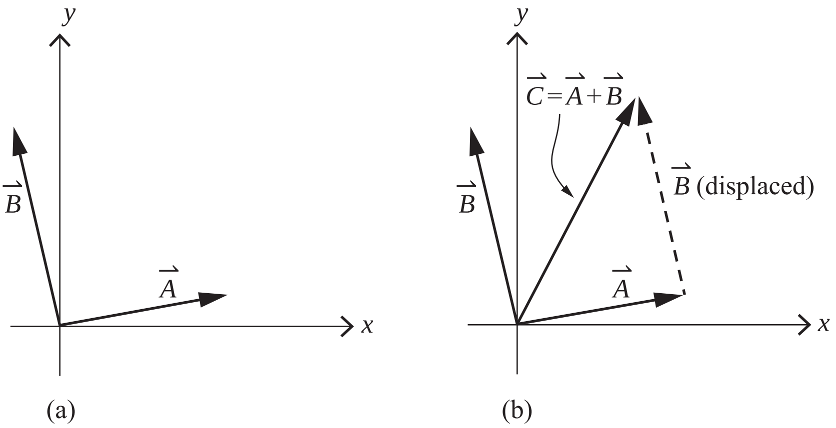

Figure 1.4 Graphical addition of vectors.

The simplest way to perform vector addition is to imagine moving one vector without changing its length or direction so that its tail (the end without the arrowhead) is at the head (the end with the arrowhead) of the other vector. The sum is then determined by making a new vector that begins at the tail of the first vector and terminates at the head of the second vector. This graphical "tail-to-head" approach to vector addition works for vectors in any direction and for three or more vectors as well.

To graphically add the two vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.4(a), imagine moving vector $\vec{B}$ without changing its length or direction so that its tail is at the position of the head of vector $\vec{A}$, as shown in Fig. 1.4(b). The sum of these two vectors is called the "resultant" vector $\vec{C} = \vec{A} + \vec{B}$; note that $\vec{C}$ extends from the tail of $\vec{A}$ to the head of $\vec{B}$. The result would have been the same had you chosen to displace the tail of vector $\vec{A}$ to the head of vector $\vec{B}$ without changing the direction of $\vec{A}$.

It's extremely important for you to note that the length of the resultant vector is *not* the length of vector $\vec{A}$ added to the length of vector $\vec{B}$ (unless $\vec{A}$ and $\vec{B}$ happen to point in the same direction). So vector addition is not the same process as scalar addition, and you should remember to never add vectors using scalar addition.

Multiplication of a vector by a scalar is also quite straightforward, because multiplying a vector by any positive scalar does not change the direction of the vector – it only scales the length of the vector. Hence, $4\vec{A}$ is a vector in exactly the same direction as $\vec{A}$, but with length four times that of $\vec{A}$, as shown in Fig. 1.5(a). If the scaling factor is less than one the resulting vector is shorter