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Introduction

Our book is devoted to the structure of the general solution of the Einstein equations with a cosmological singularity. We cover Einstein-matter systems in four and higher space-time dimensions.

Under the terminology "cosmological singularity," we mean a singularity in time, i.e., a spacelike singularity on a "submanifold" that can be viewed as the limit of a family of regular spacelike hypersurfaces forming (locally) a Gaussian foliation, such that the curvature invariants together with invariant characteristics of matter fields diverge as one tends to this submanifold.

The nonlinearities of the Einstein equations are notably known to prevent the construction of an exact general solution. From this perspective, the BKL work which describes the asymptotic *general* behavior of the gravitational field in four space-time dimensions as one approaches a spacelike singularity, is quite unique and exceptional. The central attainment of the BKL theory is the analysis of the delicate relationship between the time derivatives and the spatial gradients in the gravitational field equations near the singularity. The main technical idea of the BKL approach consists in identifying among the huge number of spatial gradients, those terms that are of the same importance as the time derivatives. In the vicinity of the singularity, these terms are in no way negligible. They act during the whole course of evolution up to the singularity, and it is actually due to these spatial gradients that oscillations do arise.

A remarkable simplifying feature nevertheless emerges as one tends to the singularity. This is the fact that the spatial gradient terms that must be retained in the dynamical equations of motion can be asymptotically represented as the products of some functions of the undifferentiated (in space) "scale factors" (which represent how distances along independent spatial directions evolve with time) by some slowly varying coefficients containing spacelike derivatives. This nontrivial separation springing up in the vicinity of the singularity leads to gravitational equations of motion which effectively reduce to a system of ordinary differential equations in time for the scale factors – one such system at each point of 3-space – because in the leading approximation, all relevant coefficients

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containing spacelike derivatives enter these equations solely as external (albeit, dynamically crucial) time-independent parameters.

Our presentation of the structure of the general solution of the Einstein equations with a cosmological singularity starts with vacuum gravity in four dimensions, where we follow the original BKL derivation. All key ideas and ingredients are already present in this model. We derive the effective dynamics and exhibit its oscillatory, chaotic behavior in Chapter 1. The effect of the "rotation of the Kasner axes," which appears in addition to the never-ending changes of Kasner exponents, is in particular carefully discussed. Some of the more technical derivations regarding this chapter are relegated to Appendix A.

The effective description of the asymptotic evolution in terms of ordinary differential equations can be reformulated as the motion of a particle in some external potential. Furthermore, it is possible to mimic the essential features of the BKL system of ordinary differential equations at any given point by considering spatially homogeneous cosmological models that have the property that their (non-abelian) homogeneity group leads to a spatial curvature for which the aforementioned dominating spatial gradients are nonzero. Such is the case for the spatially homogeneous models of so-called Bianchi types VIII and IX. The spatial curvature terms become, in the particle picture, the reflecting sharp wall potentials responsible for the oscillatory regime.

For the case of diagonal homogeneous cosmological models of Bianchi IX type, the potential has been introduced and investigated by C. Misner and D. Chitre [130, 35]. A billiard picture grew out from the work of these authors, which was very inspirational for future developments.

Chapter 2 is devoted to homogeneous cosmological models and an explanation of these developments. We also exhibit the rotation of the Kasner axes for non-diagonal spatial metrics. Appendix B provides more information about spatial homogeneity and gives, in particular, the Bianchi classification of spatially homogeneous models.

Chapter 3 discusses then the nature of the chaotic behavior near the cosmological singularity. The interesting phenomenon of "gravitational turbulence" is exhibited. Due to this phenomenon, a systematic growth of the spatial gradients arises when one approaches the singularity. At the end of this chapter we indicate that such growth does not invalidate the BKL analysis.

The BKL analysis was originally carried out for pure gravity in four space-time dimensions. However, the BKL approach can readily be extended to include matter, or by going to higher dimensions. One finds then that the same analysis of the delicate relationship between the time derivatives and the spatial gradients in the field equations near the singularity goes through, ending up again with an effective description of the dynamics at each spatial point in terms of a system of ODEs with respect to time of the "scale factors" (which also now include some of the scalar fields, if any).

What is new is that for some systems, spatial gradients are subdominant and the assertion that "only time derivatives are relevant near the singularity"

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becomes literally correct. Such singularities are "velocity-dominated" in the terminology of [66]. In some of the velocity-dominated models, the general solution is of non-oscillatory character and has simple Kasner-like power-law asymptotics near the singularity (at each spatial point). Examples are given by gravity coupled to a scalar field [16, 18], or gravity coupled to matter with a stiff equations of state, or pure gravity in space-time dimensions greater than or equal to 11 [63, 62]. In those cases, one can actually write down an explicit power-law asymptotic form of the metric valid all the way to the singularity, which contains as many arbitrary functions of space as the general solution must do, and demonstrate directly that this explicit form of the metric asymptotes an exact solution as one approaches the singularity [18, 3, 52]. This provides an independent check of the validity of the BKL procedure for such systems, which is much more explicit than for oscillatory models, where there is only the original BKL argument, which is more indirect.

These developments are first given in Chapter 4 by following the BKL approach. Perfect and viscous fluids, Yang Mills and Electromagnetic fields and scalar fields in four space-time dimensions are treated, as well as pure gravity in higher dimensions. [The general solution involving a classical spinor field is also considered because it involves interesting algebra, but because of its somewhat unphysical nature, its discussion is given in Appendix C.] We find that while the oscillatory behavior can be suppressed for some Einstein-matter models, it is present for some others. Which case arises depends on the matter content and on the space-time dimension.

The second part of our book is devoted to the billiard reformulation of the BKL behavior. Although the billiard picture originally arose in the context of homogeneous cosmological models, it is important to realize, however – and this turns out to be crucial for extensions to more general models – that the billiard motion also captures the dynamical behavior in the inhomogeneous case and is by no means tied to spatial homogeneity. While a single billiard suffices for homogeneous models, one gets one such billiard at each spatial point in the generic case. In other words, the billiard description is quite general. This is explained in Chapter 5, where we construct the billiard for pure Einstein gravity in four dimensions without any simplifying symmetry assumption. We follow the modern billiard point of view [45, 46, 51], based on the Iwasawa decomposition of the spatial metric and on a radial projection of the motion of the scale factors on the relevant hyperbolic space. On the technical side, the modern derivation streamlines the original billiard analysis by using a description in which the potential and the reflecting walls remain stationary in the vicinity of the singularity, which makes the analysis of their influence and their geometrical structure much more transparent.

The billiard viewpoint also applies to gravitational theories involving other matter fields, and in different space-time dimensions. This is explained in Chapter 6. We extend there the billiard analysis to arbitrary space-time dimensions ≥ 4, and to general systems containing gravity consistently coupled to matter

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(scalar and p-form) fields through second-order partial differential equations. We also indicate how the billiard point of view enlightens some of the results obtained in Chapter 4 through the original BKL approach.

Somewhat unexpectedly, the generalization of the billiard description gave rise to a remarkable development: it led directly to the discovery of an intriguing connection between the BKL asymptotic regime and Coxeter groups of reflections in hyperbolic space. This connection holds for the majority of theories of interest from a fundamental physical point of view, which includes pure gravity in D dimensions as well as various supergravity models [47]. It emerges because the billiard table is described in those cases by a convex polyhedron in hyperbolic space bounded by hyperplanes that make acute angles that are integer submultiples of π. This is quite remarkable in view of the fact that the angles depend on various discrete or continuous parameters: the space-time dimension, the ranks of the p-forms (if any), as well as the dilaton couplings (if dilatons are present).

For all theories of physical interest, the relevant billiard region is thus a Coxeter polyhedron (i.e., a convex polyhedron with all diehedral angles equal to integer submultiples of π). This means that it is a fundamental domain for the group of reflections in the billiard walls. The motion is a succession of such reflections, and thus defines elements of that group. But there is more. (1) The billiard table is a simplex, and this is also remarkable. Indeed, the number of walls following from the Lagrangian grows much faster than the dimension of the billiard table, but only a small subset of these walls, yielding a simplex, are dominant and relevant for the billiard description. The Coxeter group is thus a "simplex Coxeter group." (2) Furthermore, the billiard walls come with a natural normalization and define through their normalized scalar products $2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$ a matrix wich turns out to be the Cartan matrix of a Lorentzian Kac–Moody algebra. The simplex Coxeter group is thus the Weyl group of a Lorentzian Kac–Moody algebra, and the billiard region can be identified with its fundamental Weyl chamber.

These demonstrated properties have led to a conjecture that goes beyond the BKL analysis and which asserts that the corresponding infinite-dimensional Kac–Moody algebra itself might be a symmetry of an appropriate completion of the theory, the BKL Coxeter group being the signal of this huge symmetry. If true, this so-called Hidden Symmetry Conjecture would create promising new perspectives for the development of gravitation theory. However, the Hidden Symmetry Conjecture has not been proven yet, and therefore falls outside the scope of this book, which concentrates only on well-established facts. We refer to the review [51] for more information.

These intriguing developments on the connection with Coxeter groups are treated in Chapter 7, with additional information of a mathematical nature given in Appendix D.

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