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Michael Howe
Excerpt
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1
Introduction

1.1 Sources of sound

Music, calm speech, whispering leaves fluttering in a breeze are pleasant and desirable sounds. Noise, howling gales, explosions and screeching traffic are less so. A quantitative understanding of the *sources* of all such sounds can be obtained by careful analysis of the mechanical equations of motion. Most sources are very complex, frequently involving ill-defined turbulent and perhaps combusting flows and their interactions with vibrating structures, and the energy released as sound tends to be a tiny fraction of that of the structural and hydrodynamic motions. Our analysis must correctly and reliably account for this general *inefficiency* of sound generation, because small errors in source modelling can lead to very large errors in acoustic prediction.

In this book we shall consider only the simplest case in which the fluid can be regarded as continuous and locally homogeneous at all levels of subdivision. The motion of the fluid will be defined when the velocity and the thermodynamic state are specified for each of the *fluid particles* of which it may be supposed to consist. The distinctive fluid property possessed by both liquids and gases is that these fluid particles can move freely relative to one another under the influence of applied forces or other externally imposed changes at the boundaries of the fluid. Five scalar partial differential equations are required to determine these motions. They are statements of conservation of mass, momentum and energy, and they are to be solved subject to appropriate *boundary* and *initial* conditions. These equations will be used to formulate and analyse a wide range of problems; our main task will be to simplify these problems to obtain a thorough understanding of source mechanisms together with a quantitative description of the subsequent propagation of the sound including, possibly, its reflection, scattering and diffraction at solid boundaries.

A general introduction to acoustics is presented in this chapter; it forms the basis for the treatment of the fluid-structure interaction problems examined in the rest of the book. The classical acoustic wave equation is derived from the general equations of motion. Particular solutions, methods of solution and applications are then discussed, principally for situations in which acoustic sources radiate into an unbounded fluid.

1.2 Equations of motion of a fluid

The state of a fluid at time t and position $\mathbf{x} = (x_1, x_2, x_3)$ is defined when the velocity $\mathbf{v}$ and any two thermodynamic variables are specified. These quantities are governed by equations describing conservation of mass, momentum and energy, supplemented by a thermodynamic equation of state.

1.2.1 The material derivative

Let v_i denote the component of fluid velocity $\mathbf{v}$ in the x_i direction, and consider the rate at which any function $F(\mathbf{x}, t)$ varies *following the motion* of a fluid particle (a material point moving with the fluid). Let the particle be at $\mathbf{x}$ at time t, and at $\mathbf{x} + \delta\mathbf{x}$ a short time later at time $t + \delta t$, where $\delta\mathbf{x} = \mathbf{v}(\mathbf{x}, t)\delta t + \cdots$. At the new position of the fluid particle

$$F(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = F(\mathbf{x}, t) + v_j \delta t \frac{\partial F}{\partial x_j}(\mathbf{x}, t) + \delta t \frac{\partial F}{\partial t}(\mathbf{x}, t) + \cdots,$$

where the repeated suffix j implies summation over $j = 1, 2, 3$. The limiting value of $(F(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - F(\mathbf{x}, t))/\delta t$ as $\delta t \to 0$ defines the material (or 'Lagrangian') derivative DF/Dt of F:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + v_j \frac{\partial F}{\partial x_j} \equiv \frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F. \tag{1.2.1}$$

DF/Dt measures the time rate of change of F as seen by an observer moving with the fluid particle that occupies position $\mathbf{x}$ at time t.

1.2.2 Equation of continuity

A fluid particle of volume V and mass density ρ has a total mass of $\rho V \equiv (\rho V)(\mathbf{x}, t)$, where $\mathbf{x}$ denotes the position of the centroid of V at time t (Figure 1.2.1). Conservation of mass requires that $D(\rho V)/Dt = 0$, that is, that

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{V}\frac{DV}{Dt} = 0. \tag{1.2.2}$$

Now $\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\oint_S \mathbf{v} \cdot d\mathbf{S}$, where the integration is over the closed material surface S forming the boundary of V, on which the vector surface element $d\mathbf{S}$ is directed *out* of V. It is the fractional rate of increase of the volume of the

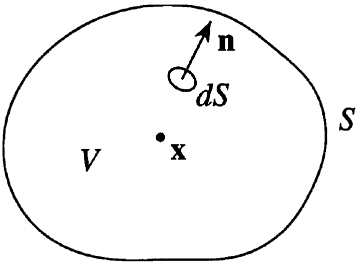

Figure 1.2.1

fluid particle, and becomes equal to div $\mathbf{v}$ as $V \to 0$. In this limit (1.2.2) can therefore be cast in any of the following equivalent forms of the *equation of continuity*

$$\left.\begin{aligned} \frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\mathbf{v} &= 0, \\ \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) &= 0, \\ \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) &= 0. \end{aligned}\right\} \tag{1.2.3}$$

1.2.3 Momentum equation

The momentum equation is derived in a similar manner by consideration of the rate of change of momentum $D(\rho V\mathbf{v})/Dt \equiv (\rho V)D\mathbf{v}/Dt$ of a fluid particle subject to pressure p and **viscous** forces acting on the surface S of V, and body forces $\mathbf{F}$ per unit volume within V. It is sufficient for our purposes to quote the form of the resulting *Navier–Stokes equation*

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p - \eta \operatorname{curl}\boldsymbol{\omega} + \left(\eta' + \frac{4}{3}\eta\right)\nabla\operatorname{div}\mathbf{v} + \mathbf{F}, \tag{1.2.4}$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ is the **vorticity**, and η and η' are respectively the **shear** and **bulk** coefficients of viscosity, which generally vary with pressure, temperature and position in the fluid. Viscous forces are important predominantly close to solid boundaries, where the frictional drag is governed by the shear viscosity η. It is then a good approximation to adopt the *Stokesian* model in which the contribution of the bulk viscosity is ignored.

Values of ρ, η and $\nu = \eta/\rho$ (the 'kinematic' viscosity) for air and water at 10°C and one atmosphere pressure are given in Table 1.2.1.

1.2.4 Energy equation

Consideration of the transfer of heat by molecular diffusion across the moving material boundary S of the fluid particle of Figure 1.2.1, and of the generation

Table 1.2.1

	ρ kg/m^3	η kg/ms	ν m^2/s
Air	1.23	1.764 10^{-5}	1.433 10^{-5}
Water	1000	1.284 10^{-3}	1.284 10^{-6}

of heat by frictional dissipation within its interior volume V leads to the energy equation

$$\rho T\frac{Ds}{Dt} = 2\eta\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)^2 + \eta' e_{kk}^2 + \operatorname{div}\left(\kappa\nabla T\right), \qquad (1.2.5)$$

where $\rho T\, Ds/Dt$ is the time rate of change following a fluid particle of the heat gained per unit volume. T, s are respectively the absolute temperature and specific entropy (i.e. entropy per unit mass), κ is the thermal conductivity of the fluid, and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (1.2.6)$$

is the rate of strain tensor, which accounts for changes in shape of the fluid particle. Because $e_{kk} \equiv \operatorname{div}\mathbf{v}$, the term involving the bulk coefficient of viscosity η' in (1.2.5) determines the dissipative production of heat during compressions and rarefactions of the fluid.

To understand the significance of the rate of strain tensor, observe that the velocity $\mathbf{v}'$ relative to the centroid of the moving fluid particle at vector distance $\mathbf{x}'$ from its centroid, is given to first order by

$$\begin{aligned} v_i' = x_j'\frac{\partial v_i}{\partial x_j} &\equiv x_j'\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + x_j'\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial x_i'}\left(e_{jk}x_j'x_k'\right) + \frac{1}{2}(\boldsymbol{\omega}\wedge\mathbf{x}')_i \end{aligned} \qquad (1.2.7)$$

where e_{ij} and the vorticity $\boldsymbol{\omega}$ are evaluated at the centroid. The term in $\boldsymbol{\omega}$ therefore represents relative *rigid body rotation* of the particle about the centroid, at angular velocity $\frac{1}{2}\boldsymbol{\omega}$, with no change of shape. The gradient term, however, represents an *irrotational* distortion of V, and is responsible for frictional forces and the conversion of mechanical energy into heat.

1.2.5 Equation of state

In the presence of velocity and pressure gradients a fluid cannot be in strict thermodynamic equilibrium, so that thermodynamic variables require special interpretation. The density ρ and the total internal energy e per unit mass can be defined in the usual way for a very small fluid particle without the need for thermodynamic equilibrium, such that ρ and ρe are the mass and internal energy per unit volume. The pressure and all other thermodynamic quantities are then defined by means of the same functions of ρ and e that would be used for a system in thermal equilibrium, and the relations between the thermodynamic variables are then the same as for a fluid in local thermodynamic equilibrium, defined by equations of the form

$$p = p(\rho, s), \quad p = p(\rho, T), \quad s = s(\rho, T), \quad \text{etc.} \tag{1.2.8}$$

The equations of state (1.2.8) permit any thermodynamic variable to be expressed in terms of any two variables, such as the density and temperature, although in applications it may be more convenient to use other such equations. There is no dissipation when sound propagates in an *ideal fluid*: $Ds/Dt = 0$ and it is usual to assume *homentropic* motion $s = s_o$ = constant. This permits the fluid motion to be determined from the equations of continuity and momentum and the equation of state $p = p(\rho, s_o)$, the energy equation being ignored. In more general situations it is necessary to retain the energy equation to account for coupling between macroscopic motions and the internal energy of the fluid.

Note, however, that the *thermodynamic pressure* $p = p(\rho, e)$ defined in this way is generally no longer the sole source of *normal stress* on any surface drawn in the fluid. This is the case in a fluid of non-zero bulk viscosity ($\eta' \neq 0$), whose molecules possess rotational (or other internal) degrees of freedom whose relaxation time required for the re-establishment of thermal equilibrium after, say, a compression, is large relative to the equilibration time of the translational degrees of freedom. When compressed (during an interval in which $\operatorname{div}\mathbf{v} < 0$) the temperature must rise, but the corresponding increase in the rotational energy lags slightly behind that of the molecular translational energy responsible for normal stress; the thermodynamic pressure p accordingly is smaller than the true normal stress (which equals $p - \eta' \operatorname{div}\mathbf{v}$). For most acoustic problems such discrepancies are sufficiently small to be neglected in a first approximation and will not be considered further.

1.2.6 Crocco's equation

The momentum equation (1.2.4) can be recast by introducing the specific enthalpy $w = e + p/\rho$ of the fluid, in terms of which the first law of thermodynamics supplies the relation

$$dw = \frac{dp}{\rho} + T\,ds. \tag{1.2.9}$$

The vector identity $(\mathbf{v}\cdot\nabla)\mathbf{v} = \boldsymbol{\omega}\wedge\mathbf{v} + \nabla\left(\frac{1}{2}v^2\right)$ and (1.2.9) permit the momentum equation to be put in **Crocco's** form

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla B = -\boldsymbol{\omega}\wedge\mathbf{v} + T\nabla s - \nu\,\mathrm{curl}\,\boldsymbol{\omega} + \left(\nu' + \tfrac{4}{3}\nu\right)\nabla\,\mathrm{div}\,\mathbf{v} + \frac{\mathbf{F}}{\rho}, \tag{1.2.10}$$

where $\nu' = \eta'/\rho$ and

$$B = w + \tfrac{1}{2}v^2 \tag{1.2.11}$$

is the **total enthalpy**. In a **perfect gas** $w = c_pT = \gamma p/(\gamma - 1)\rho$, where c_p is the specific heat at constant pressure and $\gamma = c_p/c_v$, c_v being the specific heat at constant volume.

Crocco's equation finds application in the acoustics of turbulent, heat conducting flows.

1.3 Sound waves in an ideal fluid

The intensity of an acoustic pressure p in air (relative to the mean atmospheric pressure p_o) is usually measured in decibels by the quantity

$$20 \times \log_{10}\left(\frac{|p|}{p_{ref}}\right),$$

where the reference pressure $p_{ref} = 2\times10^{-5}$ Pa. Thus, $p = p_o \equiv 1$ atmosphere ($= 10^5$ Pa) is equivalent to 194 dB. A very loud sound $\sim$ 120 dB corresponds to

$$\frac{p}{p_o} \approx \frac{2\times10^{-5}}{10^5}\times 10^{\left(\frac{120}{20}\right)} = 2\times10^{-4} \ll 1.$$

Similarly, for a 'deafening' sound of 160 dB, $p/p_o \sim 0.02$. This corresponds to a pressure of about 0.3 lbs/in^2, and is loud enough for 'nonlinear effects' to begin to be important.

The passage of a sound wave in the form of a pressure fluctuation is accompanied by a back-and-forth motion of the fluid in the direction of propagation at the *acoustic particle velocity* v, say. It will be seen (§1.3.3) that

$$\text{acoustic particle velocity} \approx \frac{\text{acoustic pressure}}{\rho_o \times \text{speed of sound}},$$

where ρ_o is the mean air density. The 'speed of sound' in air is about 340 m/sec. Thus, $v \sim 5$ cm/sec at 120 dB; at 160 dB $v \sim 5$ m/sec.

1.3.1 The wave equation for an ideal fluid

In most applications the acoustic pressure is very small relative to p_o, and sound propagation is studied by linearising the equations of motion. We consider first the simplest case of sound propagation in an *ideal fluid* – that is, a homogeneous, inviscid, non–heat-conducting fluid of mean pressure p_o and density ρ_o, which is at rest in the absence of the sound. The energy equation (1.2.5) implies that $Ds/Dt = 0$ so that sound propagation is homentropic (*adiabatic*) with $s = s_o \equiv s(p_o, \rho_o) =$ constant throughout fluid. The implication is that, in an ideal fluid there is negligible dissipation of the organised mechanical energy of the sound by heat and momentum transfer by molecular diffusion between neighbouring fluid particles.

The departures of the pressure and density from their undisturbed values are denoted by p', ρ' where $p'/p_o \ll 1$, $\rho'/\rho_o \ll 1$. The linearised form of the momentum equation (1.2.4) for an ideal fluid ($\eta = \eta' = 0$) then becomes

$$\rho_o \frac{\partial \mathbf{v}}{\partial t} + \nabla p' = \mathbf{F}. \tag{1.3.1}$$

Before linearising the continuity equation (1.2.3) it is useful to make an artificial generalisation by inserting a **volume source** distribution $q(\mathbf{x}, t)$ on the right-hand side:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div} \mathbf{v} = q; \tag{1.3.2}$$

q is the rate of increase of fluid volume per unit volume of the fluid, and might represent, for example, the effect of volume pulsations of a small body in the fluid (§1.4). The linearised equation is then

$$\frac{1}{\rho_o}\frac{\partial \rho'}{\partial t} + \operatorname{div} \mathbf{v} = q. \tag{1.3.3}$$

Eliminate $\mathbf{v}$ between (1.3.1) and (1.3.3):

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = \rho_o \frac{\partial q}{\partial t} - \operatorname{div} \mathbf{F}. \tag{1.3.4}$$

An equation determining the pressure p' alone in terms of q and $\mathbf{F}$ is obtained by invoking the homentropic approximation $p = p(\rho, s_o)$, where $p_o = p(\rho_o, s_o)$ in the undisturbed state. Therefore

$$p_o + p' = p(\rho_o + \rho', s_o) \approx p(\rho_o, s_o) + \rho' \frac{\partial p}{\partial \rho}(\rho, s_o), \tag{1.3.5}$$

where the derivative is evaluated at the undisturbed value ρ_o of the density. It has the dimensions of (velocity)^2 and its square root defines the *speed of sound*

$$c_o = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \tag{1.3.6}$$

where differentiation is performed at $s = s_o$, and evaluated at $\rho = \rho_o$. In air $c_o \approx 340$ m/s; in water $c_o \approx 1500$ m/s.

From (1.3.5): $\rho' = p'/c_o^2$, and substitution for ρ' in (1.3.4) yields the inhomogeneous wave equation

$$\left(\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p = \rho_o \frac{\partial q}{\partial t} - \operatorname{div} \mathbf{F}, \tag{1.3.7}$$

where the prime $'$ on the acoustic pressure has been discarded. This equation governs the production of sound waves by the volume source q and the force $\mathbf{F}$. When these terms are absent the equation describes sound propagation from sources on the boundaries of the fluid, such as the vibrating cone of a loudspeaker.

The volume source q and (with the exception of gravity) the body force $\mathbf{F}$ would never appear in a complete description of sound generation in a real fluid. They are introduced only when we *think* we understand how to 'model' mathematically the real sources of sound in terms of idealised volume sources and forces. But this can be dangerous and misleading – small errors in specifying the sources often result in very large errors in the predicted sound, because only a tiny fraction of the available energy of a vibrating fluid or structure actually radiates away as sound.

When $\mathbf{F} = \mathbf{0}$ equation (1.3.1) implies the existence of a velocity potential φ such that $\mathbf{v} = \nabla\varphi$, in terms of which the perturbation pressure is given by

$$p = -\rho_o \frac{\partial \varphi}{\partial t}. \tag{1.3.8}$$

It follows from this and (1.3.7) (with $\mathbf{F} = \mathbf{0}$) that the velocity potential is the solution of

$$\left(\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \varphi = -q(\mathbf{x}, t), \tag{1.3.9}$$

where causality has been invoked to discard any time-independent constants of integration in passing from (1.3.7) to (1.3.9). Equation (1.3.9) is the wave equation of classical acoustics.

In the 'propagation zone,' where the source terms $q = 0$, $\mathbf{F} = \mathbf{0}$, the velocity $\mathbf{v}$ and the perturbations in p, ρ (and in other thermodynamic quantities such as the temperature T, internal energy e and enthalpy w, but *not* the specific entropy

Table 1.3.1 *Speed of sound and acoustic wavelength*

	c_o				λ at 1 kHz	
	m/s	f/s	km/h	mph	metres	feet
Air	340	1100	1225	750	0.3	1
Water	1500	5000	5400	3400	1.5	5

s, which remains constant and equal to s_o) propagate as sound governed by the homogeneous form of (1.3.9). The velocity fluctuation $\mathbf{v}$ produced by the passage of the wave is the **acoustic particle velocity**.

1.3.2 Speed of sound

In a perfect gas $p = \rho \mathrm{R} T$ and $s = c_v \ln(p/\rho^\gamma)$, where $\mathrm{R} = c_p - c_v$ is the gas constant, and in the linearised approximation

$$c_o = \sqrt{\gamma p_o/\rho_o} = \sqrt{\gamma \mathrm{R} T_o}. \tag{1.3.10}$$

The wavelength λ of sound of frequency f Hz is the distance travelled by the sound in one period $1/f$, that is, $\lambda = c_o/f$. Typical approximate speeds of sound in air and in water, and the corresponding acoustic wavelength at $f = 1$ kHz are given in Table 1.3.1.

1.3.3 Plane waves

A plane acoustic wave propagating in the x direction satisfies

$$\left(\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\varphi = 0, \tag{1.3.11}$$

which has the general solution (D'Alembert 1747)

$$\varphi = \Phi\left(t - \frac{x}{c_o}\right) + \Psi\left(t + \frac{x}{c_o}\right), \tag{1.3.12}$$

where $\Phi,\ \Psi$ are arbitrary functions that respectively represent waves travelling at speed c_o without change of form in the positive and negative x directions. The acoustic particle velocity $\mathbf{v} = \nabla\varphi$ is parallel to the propagation direction (the waves are 'longitudinal').

The solutions (1.3.12) and the linearised (source-free) equations of motion can be used to show that fluctuations in $v,\ p,\ \rho',\ T'$ and w' in a plane wave propagating parallel to the x-axis in a perfect gas are related by

$$v = \pm\frac{p}{\rho_o c_o}, \quad \rho' = \frac{p}{c_o^2}, \quad T' = \frac{p}{\rho_o c_p}, \quad w' = \frac{p}{\rho_o}, \tag{1.3.13}$$

where v is measured in the positive x direction and the $\pm$ sign is taken according as the wave propagates in the positive or negative x direction.

Example 1 Waves in a uniform tube generated by an oscillating piston The end $x = 0$ of an infinitely long, uniform tube is closed by a smoothly sliding piston executing small amplitude normal oscillations at velocity $u_o(t)$ (Figure 1.3.1a). If x increases along the tube, linear acoustic theory and the requirement that sound waves must propagate to $x = +\infty$, away from the piston, imply that $p = p(t - x/c_o)$. At $x = 0$ the velocities of the fluid and piston are the same, so that

$$u_o(t) \equiv \frac{p(t)}{\rho_o c_o}, \qquad \therefore \qquad p = \rho_o c_o u_o(t - x/c_o) \ \text{ for } \ x > 0.$$

In practice a solution of this kind, where energy is confined by the tube to propagate in waves of constant cross-section, becomes progressively invalid as x increases, because of the accumulation of small effects of flow nonlinearity. Nonlinear analysis reveals that at a point in the wave where the particle velocity is v the wave actually propagates at speed $c_o + v$, so that wave elements where v is large and positive produce 'wave steepening,' resulting ultimately in the formation of 'shock waves.' This type of behaviour is important, for example, for waves generated in a long railway tunnel by the piston effect of an entering high-speed train.

Example 2 Reflection at a closed end (Figure 1.3.1b) Let the plane wave $p = p_I(t - x/c_o)$ approach from $x < 0$ the closed, rigid end at $x = 0$ of a uniform, semi-infinite tube. The reflected pressure $p_R(t + x/c_o)$ is determined by the condition that the (normal) fluid velocity must vanish at $x = 0$. Therefore, $p_I(t)/\rho_o c_o - p_R(t)/\rho_o c_o = 0$, and the overall pressure within the tube is given by

$$p = p_I(t - x/c_o) + p_I(t + x/c_o), \quad x < 0.$$

Reflection at the rigid end causes 'pressure doubling at the wall,' where $p = 2p_I(t)$.

Example 3 Reflection at an open end (Figure 1.3.1c) When the wavelength of the sound is large compared to the radius R of an open ended circular cylindrical tube, the first approximation to the condition satisfied by the acoustic pressure at the open end ($x = 0$) is that the overall pressure