1 Introduction

Quantum theory is the soul of contemporary physics. It was discovered in an adventurous way, under the urge to solve the puzzles posed by atomic spectra and blackbody radiation. But after its invention, it immediately became clear that it was not just a theory of specific physical systems: it was rather a new language of universal applicability. Already in 1928, the theory had received solid mathematical foundations by Hilbert, von Neumann, and Nordheim,[1] and this work was brought to completion in the monumental work of von Neumann,[2] in the form that we still study nowadays. The theory is extraordinarily successful, and its predictions have been confirmed to an astonishing level of precision in a large spectrum of experiments.

However, almost 90 years after von Neumann's book, quantum theory remains mysterious. Its mathematical formulation – based on Hilbert spaces and self-adjoint operators – is far from having an intuitive interpretation. The association of physical systems to Hilbert spaces whose unit vectors represent pure states, the representation of transformations by unitary operators and of observables by self-adjoint operators – all such postulates look artificial and *ad hoc*. A slightly more operational approach is provided by the C^*-algebraic formulation of quantum theory[3] – still, this formulation relies on the assumption that observables form an *algebra*, where the physical meanings of the multiplication and the sum are far from clear.

In short, the postulates of quantum theory impose mathematical structures without providing any simple reason for this choice: the mathematics of Hilbert spaces is adopted as a magic blackbox that "works well" at producing experimental predictions. However, in a satisfactory axiomatization of a physical theory the mathematical structures should emerge as a consequence of postulates that have a direct physical interpretation. By this we mean postulates referring, e.g., to primitive notions like physical system, measurement, or process, rather than notions like, e.g., Hilbert space, C^*-algebra, unit vector, or self-adjoint operator.

The crucial question thus remains unanswered: why quantum theory? Which are the principles at the basis of the theory? A case that is often invoked in contrast is that of Special Relativity theory, which directly follows from the simple understandable principle of relativity.

[1] Hilbert *et al.* (1928).
[2] The book (von Neumann, 1932) has been recently reprinted (von Neumann, 1996).
[3] Haag (1993).

1.1 The Quest for Principles: von Neumann

The need for a deeper understanding of quantum theory in terms of fundamental principles was clear since the very beginning. Von Neumann himself expressed his dissatisfaction with his own mathematical formulation of quantum theory with the surprising words, “I don’t believe in Hilbert space anymore.”[4] Realizing the physical relevance of the axiomatization problem, Birkhoff and von Neumann made an attempt at understanding quantum theory as a new kind of *propositional calculus*,[5] motivated by the opinion that the main difficulties in accepting the “quantumness” of elementary physical systems stem from the inadequacy of classical logic to encompass the unpredictable nature of quantum measurement outcomes. In their attempt, von Neumann and Birkhoff proposed to treat the propositions about the physical world in a suitable logical framework, different from classical logic, where the operations AND and OR are no longer distributive. The lack of interpretation of the observables algebra led Jordan, von Neumann, and Wigner to consider the possibility of a commutative algebra of observables, with a product that only requires the definition of squares and sums of observables – the so-called *Jordan product*.[6] These works inaugurated the tradition of quantum logics, which led to several attempts at an axiomatization of quantum theory, most notably by Mackey,[7] Varadarajan,[8] and Jauch and Piron,[9, 10] with ramifications still the object of active research.[11]

Researchers in quantum logic managed to derive a significant part of the quantum framework from logical axioms. In general, a certain degree of technicality (mainly related to the emphasis on infinite-dimensional systems) makes these results far from providing a clear-cut description of quantum theory in terms of fundamental principles. Even among the experts there is a general consensus that the axioms are not as insightful as one would have hoped. For both experts and non-experts, it is hard to figure out what is the moral of the quantum logic axiomatizations: what is special about quantum theory after all? Why should quantum theory be preferred to alternative theories?

A notable alternative axiomatization program was that of Ludwig,[12] who adopted an operational approach, where the basic notions are those of preparation and measuring devices, and the postulates specify how preparations and measurements combine to give the probabilities of experimental outcomes. However, even Ludwig’s program never succeed in deriving Hilbert spaces from operational principles, as some of the postulates still contained mathematical notions with no operational interpretation.

4 This was reported by Birkhoff (1984).
5 Birkhoff and von Neumann (1936).
6 Jordan *et al.* (1934). See the recent encyclopedic books of Alfsen and Shultz (2001, 2003).
7 Mackey (1963).
8 Varadarajan (1962).
9 Jauch and Piron (1963); Piron (1964, 1976). Foulis and Randall developed an empirical counterpart of Piron’s approach (Foulis *et al.*, 1983; Foulis and Randall, 1984).
10 For a thorough textbook see Beltrametti *et al.* (2010).
11 For a review on the more recent progresses of quantum logics see Coecke *et al.* (2000).
12 Ludwig (1983).

1.2 Quantum Information Resurrects the Quest

The ambition to find a more insightful axiomatization re-emerged with the rise of quantum information. The new field showed that the mathematical axioms of quantum theory imply striking operational consequences, such as quantum key distribution,[13] quantum algorithms,[14] no cloning,[15] quantum teleportation,[16] and dense coding.[17] A natural question is then: can we reverse the implication and *derive* the mathematics of quantum theory from some of its operational features? This question lies at the core of a research program launched by Fuchs[18] and Brassard,[19] which can be synthesized by the motto "quantum foundations in the light of quantum information."[20] The ultimate goal of the program is to reconstruct the whole structure of quantum theory from a few simple principles of information-theoretic nature.

One may wonder why quantum information theorists should be more successful than their predecessors in the axiomatic endeavor. A good reason is the following. In the pre-quantum information era, quantum theory was viewed like an impoverished version of classical theory, lacking the ability to make deterministic predictions about the outcomes of experiments. Clearly, this perspective offered no vantage point for explaining why the world should be quantum. Contrarily, quantum information provided plenty of positive reasons for preferring quantum theory to its classical counterpart – as many good reasons as the number of useful quantum information and computation protocols. Turning some of these reasons into axioms then appeared as a promising route towards a compelling axiomatization.[21]

The quantum information approach can also be regarded as an evolution of the quantum logic program, where quantum theory – rather than being considered as an alternate logical system – is regarded as an alternate theory of information processing, namely describing information sources and information-processing channels. Indeed, in classical probability theory, logic can be regarded as the special case of information-processing theory where the probabilities of events are bound to the truth values $\{0, 1\}$. In non-deterministic theories like quantum theory, however, there are events whose truth value cannot be assessed, and one must concede that all we know about them is their occurrence probability.

Another new feature of the quantum information approach has been to shift the emphasis to finite-dimensional systems, which allow for a simpler treatment but still possess all the

13 Wiesner (1983); Bennett *et al.* (1984); Ekert (1991).
14 Grover (1996); Shor (1997).
15 Dieks (1982); Wootters and Zurek (1982).
16 Bennett *et al.* (1993).
17 Bennett and Wiesner (1992).
18 Fuchs (2002, 2003).
19 Brassard (2005).
20 Fuchs *et al.* (2001). This was also the title of one influential conference, held in May 2000 at the Université de Montréal, which kickstarted the new wave of quantum axiomatizations.
21 See Clifton *et al.* (2003). This work, however, assumed a C^*-algebra framework, and used informational-theoretical constraints for selecting the algebra, in particular for adopting the quantum versus the classical algebra.

relevant quantum features. In a sense, the study of finite-dimensional systems allows one to decouple the conceptual difficulties in our understanding of quantum theory from the technical difficulties of infinite-dimensional systems.

In this scenario, Hardy in 2001[22] reopened the debate about axiomatization with fresh ideas resting on the quantum information experience. Some of his axioms, however, contained mathematical notions with no interpretation, e.g. statements about the dimensionality of the state space, or the continuity of the set of pure states. Stimulated by Hardy's and Fuchs's works one of the authors of this book addressed a new axiomatization approach[23] based on operational principles about tomography, calibration and composition of transformations, and generally on the reduction of experimental complexity, such as the existence of a pure faithful state, a property that allows for tomography of transformations preparing a single input pure state. However, a thorough derivation of the theory was still missing, and also in this case there remained mathematical postulates with no interpretation. Later, building on Hardy's work the program flourished, leading to an explosion of new axiomatizations based on a variety of conceptual and mathematical frameworks,[24,25] including the framework and axiomatization contained in the present book.[26] These works realized the old dream of Wheeler's program "it from bit," for which he argued that "all things physical are information-theoretic in origin."[27]

1.3 Quantum Theory as an OPT

A lesson that we learned from the experience of quantum information is to regard quantum theory as a theory of information processing in the first place. We thus realized the crucial role played by the description of processes in the form of quantum circuits. This has led us to consider quantum theory as an extension of probability theory, to which we add the crucial ingredient of *connectivity* among events. This means that to the joint events we associate not only their joint probability, but also a circuit that connects them. When the events in the circuit have a well-defined order, the circuit is mathematically described by a *directed acyclic graph* (a graph with directed edges and without loops). Therefore, if we want to predict a joint probability, the varibles to be specified are not only the events but also the circuit connecting them.

A theory for making predictions about joint events depending on their reciprocal connections is what we call an *operational probabilistic theory* (OPT). We see that OPT is a non-trivial extension of probability theory, which, according to Jaynes and Cox,[28] in turn

22 Hardy (2001).
23 D'Ariano (2006a,b, 2007a,b, 2010); D'Ariano and Tosini (2010).
24 Goyal *et al.* (2010); Dakic and Brukner (2011); Hardy (2011); Masanes and Müller (2011); Masanes *et al.* (2013); Wilce (2012); Barnum *et al.* (2014).
25 For a comprehensive collection of papers see the book by Chiribella and Spekkens (2015).
26 Chiribella *et al.* (2010a, 2011).
27 Wheeler (1990).
28 Jaynes (2003); Cox (1961).

is an extension of logic.[29] We now realize how, in the previous axiomatization attempts, only one facet of quantum theory was considered, consisting of propositional calculus and probability, whereas the connectivity facet was missing.

From what we have said, we now understand how the basic element of an OPT – the notion of *event* – gets dressed with *wires* that allow us to connect it with other events. Such wires are the *systems* of the theory. In agreement with the directed nature of the graph, there are *input* and *output* systems. The events are the *transformations*, whereas the transformations with no input system are the *states* (corresponding to preparations of systems), and those with no output system are the *effects* (corresponding to observations of systems). Since the purpose of a single event is to describe a process connecting an input with an output, the full circuit associated to a probability is a closed one, namely a circuit with no input and no output.

The circuit framework is mathematically formalized in the language of *category theory*.[30] In this language, an OPT is a category, whose systems and events are *objects* and *arrows*, respectively. Every arrow has an input and an output object, and arrows can be sequentially composed. The associativity, existence of a trivial system, and commutativity of the parallel composition of systems of quantum theory technically correspond to having a *strict symmetric monoidal category*.[31] Although the OPT language can be rephrased in purely category theoretical terms, its original version[32] is more physicist-friendly, and it will be adopted in the present book. Expressions in such a language have an immediate meaning as the description of elementary physical processes and their relations within an experimental setting – for example, specifying whether two events occur in sequence or in parallel. However, we note the indispensable role of the probabilistic structure in promoting the OPT language from a merely descriptive tool to a framework for prediction, which is the crucial feature of a scientific theory. Two OPTs will then be different if they have different rules for assigning probabilities to the circuits.

1.4 The Principles

OPTs provide a general unified framework to formalize theories of information, including classical information theory and quantum information theory. In this framework, we characterize quantum theory *as a theory of information*. In short, quantum theory is the theory which allows for the optimal validation of randomness: all the six principles of the theory come together in such respect from complementary standpoints. Five of the six principles – causality, local discriminability, perfect discriminability, ideal compression, and atomicity of composition – express ordinary properties that are shared by quantum and

[29] We would like to mention the famous quote of J. C. Maxwell: “the true logic for this world is the calculus of probabilities.” See also Keynes (2004).
[30] Mac Lane (1978).
[31] For an introduction to the graphical language of monoidal catagories we recommend the beautiful surveys by Selinger (n.d.) and Coecke (2008).
[32] The language of OPTs was introduced in Chiribella *et al.* (2010a).

classical information theory. The sixth principle – purification – identifies quantum theory uniquely.

In non-technical words, the six principles are the following:

- **Causality**. Measurement results cannot depend on what is done on the system at the output of the measurement. Equivalently: no signal can be sent from the future to the past.
- **Local discriminability**. We can reconstruct the joint state of multiple systems by performing only local measurements on each system.
- **Perfect discriminability**. Every state that is not completely mixed can be perfectly distinguished from some other state.
- **Ideal compression**. Every source of information[33] can be encoded in a lossless and maximally efficient fashion (*lossless* means perfectly decodable, *maximally efficient* means that every state of the encoding system represents a state of the source).
- **Atomicity of composition**. No side information can hide in the composition of two atomic transformations. Equivalently: the sequential composition of two precisely known transformations is precisely known.
- **Purification**. Every random preparation of a system can be achieved by a pure preparation of the system with an environment, in a way that is essentially unique.

The first five principles of the list are satisfied by classical information theory. Hence, in our axiomatization, the purification principle is highlighted as the distinctive axiom of quantum theory. All the six principles have an epistemological nature. Causality is necessary for control of observations, shielding them from the influence of external agents acting in the future or from far apart. Local discriminability allows for the local accessibility of information. Perfect discriminability allows for falsifiability of propositions of the theory. Atomicity of composition allows for control in composing transformations and observations. Purification allows for validation of randomness, by leaving to an agent access to both system and environment.

It is important to remark here the value of the six principles for philosophy of science. For example the local discriminability principle reconciles the holism of a theory with the reductionistic approach, as explained in Chapter 6. Paradigmatic is the principle of causality, which would be matter for a treatise, in consideration of the wealth of literature on the subject in philosophy and physics. To realize the subtlety of the notion one can just consider the simple fact that causality has never been formally stated as a principle in physics.[34] Mostly the causality notion has been misunderstood due to a spurious connection with the independent notion of determinism.[35] The causality principle for quantum theory is the logical quintessence of the meaningful notions debated within the

[33] An information source technically is a set of states of a fixed system.

[34] Only very recently it has been explicitly remarked by some authors that causality is built in quantum theory (Ellis, 2008).

[35] The logical independence between the notion of causality and that of determinism is proved by the existence of causal OPTs that are not deterministic, e.g. quantum theory, and vice versa of deterministic theories that are not causal, as those constructed in D'Ariano *et al.* (2014a).

specialized literature since Hume, and ranging to modern and contemporary authors.[36] The language of OPT provides the right framework for formalizing the notion of causality in a theory-independent manner, offering a rigorous notion for philosophical analysis. Such notion also corresponds to the standard use of causality in inference and scientific modeling,[37] and coincides with the Einstenian causality, as explained in Chapter 5.

The purification principle is also of great relevance for philosophy of science. It is the axiom that selects quantum theory, thus containing its essence. Its conceptual content is the expression of a law of *conservation of information*, stating that irreversibility is in principle reducible to a lack of control over an environment. More precisely, the principle is equivalent to stating that every irreversible process can be simulated in an essentially unique way by a reversible interaction of the system with an environment, initially prepared in a pure state.[38] This statement includes the case of measurement processes, and in that case it implies the possibility of arbitrarily shifting the cut between the observer and the observed system. The arbitrariness of such a shift was considered by von Neumann as a “fundamental requirement of the scientific viewpoint,”[39] and his discussion of the measurement process was exactly aimed at showing that quantum theory fulfills it. Finally, the principle of purification is of great relevance for philosophy of probability,[40] since it provides the existence of random sources that can be validated by a measurement performed jointly on the source and on the purifying system.

36 Salmon (1967); Dowe (2007).
37 Pearl (2012).
38 Chiribella *et al.* (2010a).
39 See p. 418 of von Neumann (1996).
40 Gillies (2000).