
Contents

| <i>Preface</i> | <i>page xi</i> |
|---|----------------|
| 1 Nonparametric Statistical Models | 1 |
| 1.1 Statistical Sampling Models | 2 |
| 1.1.1 Nonparametric Models for Probability Measures | 2 |
| 1.1.2 Indirect Observations | 3 |
| 1.2 Gaussian Models | 4 |
| 1.2.1 Basic Ideas of Regression | 4 |
| 1.2.2 Some Nonparametric Gaussian Models | 6 |
| 1.2.3 Equivalence of Statistical Experiments | 8 |
| 1.3 Notes | 13 |
| | |
| 2 Gaussian Processes | 15 |
| 2.1 Definitions, Separability, 0-1 Law, Concentration | 15 |
| 2.1.1 Stochastic Processes: Preliminaries and Definitions | 15 |
| 2.1.2 Gaussian Processes: Introduction and First Properties | 19 |
| 2.2 Isoperimetric Inequalities with Applications to Concentration | 26 |
| 2.2.1 The Isoperimetric Inequality on the Sphere | 26 |
| 2.2.2 The Gaussian Isoperimetric Inequality for the Standard Gaussian Measure on \mathbb{R}^N | 30 |
| 2.2.3 Application to Gaussian Concentration | 32 |
| 2.3 The Metric Entropy Bound for Suprema of Sub-Gaussian Processes | 36 |
| 2.4 Anderson's Lemma, Comparison and Sudakov's Lower Bound | 48 |
| 2.4.1 Anderson's Lemma | 48 |
| 2.4.2 Slepian's Lemma and Sudakov's Minorisation | 52 |
| 2.5 The Log-Sobolev Inequality and Further Concentration | 60 |
| 2.5.1 Some Properties of Entropy: Variational Definition and Tensorisation | 60 |
| 2.5.2 A First Instance of the Herbst (or Entropy) Method: Concentration of the Norm of a Gaussian Variable about Its Expectation | 62 |
| 2.6 Reproducing Kernel Hilbert Spaces | 66 |
| 2.6.1 Definition and Basic Properties | 66 |
| 2.6.2 Some Applications of RKHS: Isoperimetric Inequality, Equivalence and Singularity, Small Ball Estimates | 72 |
| 2.6.3 An Example: RKHS and Lower Bounds for Small Ball Probabilities of Integrated Brownian Motion | 79 |
| 2.7 Asymptotics for Extremes of Stationary Gaussian Processes | 88 |
| 2.8 Notes | 102 |

| | | |
|----------|--|-----|
| 3 | Empirical Processes | 109 |
| 3.1 | Definitions, Overview and Some Background Inequalities | 109 |
| 3.1.1 | Definitions and Overview | 109 |
| 3.1.2 | Exponential and Maximal Inequalities for Sums of Independent Centred and Bounded Real Random Variables | 113 |
| 3.1.3 | The Lévy and Hoffmann-Jørgensen Inequalities | 121 |
| 3.1.4 | Symmetrisation, Randomisation, Contraction | 127 |
| 3.2 | Rademacher Processes | 135 |
| 3.2.1 | A Comparison Principle for Rademacher Processes | 136 |
| 3.2.2 | Convex Distance Concentration and Rademacher Processes | 139 |
| 3.2.3 | A Lower Bound for the Expected Supremum of a Rademacher Process | 144 |
| 3.3 | The Entropy Method and Talagrand's Inequality | 149 |
| 3.3.1 | The Subadditivity Property of the Empirical Process | 149 |
| 3.3.2 | Differential Inequalities and Bounds for Laplace Transforms of Subadditive Functions and Centred Empirical Processes, $\lambda \geq 0$ | 153 |
| 3.3.3 | Differential Inequalities and Bounds for Laplace Transforms of Centred Empirical Processes, $\lambda < 0$ | 158 |
| 3.3.4 | The Entropy Method for Random Variables with Bounded Differences and for Self-Bounding Random Variables | 161 |
| 3.3.5 | The Upper Tail in Talagrand's Inequality for Nonidentically Distributed Random Variables* | 165 |
| 3.4 | First Applications of Talagrand's Inequality | 171 |
| 3.4.1 | Moment Inequalities | 171 |
| 3.4.2 | Data-Driven Inequalities: Rademacher Complexities | 173 |
| 3.4.3 | A Bernstein-Type Inequality for Canonical U -statistics of Order 2 | 175 |
| 3.5 | Metric Entropy Bounds for Suprema of Empirical Processes | 184 |
| 3.5.1 | Random Entropy Bounds via Randomisation | 184 |
| 3.5.2 | Bracketing I: An Expectation Bound | 195 |
| 3.5.3 | Bracketing II: An Exponential Bound for Empirical Processes over Not Necessarily Bounded Classes of Functions | 206 |
| 3.6 | Vapnik-Červonenkis Classes of Sets and Functions | 212 |
| 3.6.1 | Vapnik-Červonenkis Classes of Sets | 212 |
| 3.6.2 | VC Subgraph Classes of Functions | 217 |
| 3.6.3 | VC Hull and VC Major Classes of Functions | 222 |
| 3.7 | Limit Theorems for Empirical Processes | 228 |
| 3.7.1 | Some Measurability | 229 |
| 3.7.2 | Uniform Laws of Large Numbers (Glivenko-Cantelli Theorems) | 233 |
| 3.7.3 | Convergence in Law of Bounded Processes | 242 |
| 3.7.4 | Central Limit Theorems for Empirical Processes I: Definition and Some Properties of Donsker Classes of Functions | 250 |
| 3.7.5 | Central Limit Theorems for Empirical Processes II: Metric and Bracketing Entropy Sufficient Conditions for the Donsker Property | 257 |
| 3.7.6 | Central Limit Theorems for Empirical Processes III: Limit Theorems Uniform in P and Limit Theorems for P -Pre-Gaussian Classes | 261 |
| 3.8 | Notes | 286 |

Contents

ix

| | | |
|----------|---|-----|
| 4 | Function Spaces and Approximation Theory | 291 |
| 4.1 | Definitions and Basic Approximation Theory | 291 |
| 4.1.1 | Notation and Preliminaries | 291 |
| 4.1.2 | Approximate Identities | 295 |
| 4.1.3 | Approximation in Sobolev Spaces by General Integral Operators | 301 |
| 4.1.4 | Littlewood-Paley Decomposition | 304 |
| 4.2 | Orthonormal Wavelet Bases | 305 |
| 4.2.1 | Multiresolution Analysis of L^2 | 305 |
| 4.2.2 | Approximation with Periodic Kernels | 312 |
| 4.2.3 | Construction of Scaling Functions | 316 |
| 4.3 | Besov Spaces | 327 |
| 4.3.1 | Definitions and Characterisations | 327 |
| 4.3.2 | Basic Theory of the Spaces B_{pq}^s | 338 |
| 4.3.3 | Relationships to Classical Function Spaces | 347 |
| 4.3.4 | Periodic Besov Spaces on $[0, 1]$ | 352 |
| 4.3.5 | Boundary-Corrected Wavelet Bases* | 361 |
| 4.3.6 | Besov Spaces on Subsets of \mathbb{R}^d | 366 |
| 4.3.7 | Metric Entropy Estimates | 372 |
| 4.4 | Gaussian and Empirical Processes in Besov Spaces | 379 |
| 4.4.1 | Random Gaussian Wavelet Series in Besov Spaces | 379 |
| 4.4.2 | Donsker Properties of Balls in Besov Spaces | 381 |
| 4.5 | Notes | 386 |
| 5 | Linear Nonparametric Estimators | 389 |
| 5.1 | Kernel and Projection-Type Estimators | 389 |
| 5.1.1 | Moment Bounds | 391 |
| 5.1.2 | Exponential Inequalities, Higher Moments and Almost-Sure Limit Theorems | 405 |
| 5.1.3 | A Distributional Limit Theorem for Uniform Deviations* | 411 |
| 5.2 | Weak and Multiscale Metrics | 421 |
| 5.2.1 | Smoothed Empirical Processes | 421 |
| 5.2.2 | Multiscale Spaces | 434 |
| 5.3 | Some Further Topics | 439 |
| 5.3.1 | Estimation of Functionals | 439 |
| 5.3.2 | Deconvolution | 451 |
| 5.4 | Notes | 462 |
| 6 | The Minimax Paradigm | 467 |
| 6.1 | Likelihoods and Information | 467 |
| 6.1.1 | Infinite-Dimensional Gaussian Likelihoods | 468 |
| 6.1.2 | Basic Information Theory | 473 |
| 6.2 | Testing Nonparametric Hypotheses | 476 |
| 6.2.1 | Construction of Tests for Simple Hypotheses | 478 |
| 6.2.2 | Minimax Testing of Uniformity on $[0, 1]$ | 485 |
| 6.2.3 | Minimax Signal-Detection Problems in Gaussian White Noise | 492 |
| 6.2.4 | Composite Testing Problems | 494 |
| 6.3 | Nonparametric Estimation | 511 |
| 6.3.1 | Minimax Lower Bounds via Multiple Hypothesis Testing | 512 |

| | | |
|----------|--|------------|
| 6.3.2 | Function Estimation in L^∞ Loss | 515 |
| 6.3.3 | Function Estimation in L^p -Loss | 518 |
| 6.4 | Nonparametric Confidence Sets | 522 |
| 6.4.1 | Honest Minimax Confidence Sets | 523 |
| 6.4.2 | Confidence Sets for Nonparametric Estimators | 524 |
| 6.5 | Notes | 537 |
| 7 | Likelihood-Based Procedures | 541 |
| 7.1 | Nonparametric Testing in Hellinger Distance | 542 |
| 7.2 | Nonparametric Maximum Likelihood Estimators | 546 |
| 7.2.1 | Rates of Convergence in Hellinger Distance | 547 |
| 7.2.2 | The Information Geometry of the Likelihood Function | 551 |
| 7.2.3 | The Maximum Likelihood Estimator over a Sobolev Ball | 554 |
| 7.2.4 | The Maximum Likelihood Estimator of a Monotone Density | 563 |
| 7.3 | Nonparametric Bayes Procedures | 570 |
| 7.3.1 | General Contraction Results for Posterior Distributions | 573 |
| 7.3.2 | Contraction Results with Gaussian Priors | 578 |
| 7.3.3 | Product Priors in Gaussian Regression | 582 |
| 7.3.4 | Nonparametric Bernstein–von Mises Theorems | 591 |
| 7.4 | Notes | 603 |
| 8 | Adaptive Inference | 607 |
| 8.1 | Adaptive Multiple-Testing Problems | 607 |
| 8.1.1 | Adaptive Testing with L^2 -Alternatives | 608 |
| 8.1.2 | Adaptive Plug-in Tests for L^∞ -Alternatives | 612 |
| 8.2 | Adaptive Estimation | 614 |
| 8.2.1 | Adaptive Estimation in L^2 | 614 |
| 8.2.2 | Adaptive Estimation in L^∞ | 620 |
| 8.3 | Adaptive Confidence Sets | 628 |
| 8.3.1 | Confidence Sets in Two-Class Adaptation Problems | 629 |
| 8.3.2 | Confidence Sets for Adaptive Estimators I | 638 |
| 8.3.3 | Confidence Sets for Adaptive Estimators II: Self-Similar Functions | 644 |
| 8.3.4 | Some Theory for Self-Similar Functions | 657 |
| 8.4 | Notes | 664 |
| | <i>References</i> | 667 |
| | <i>Author Index</i> | 683 |
| | <i>Index</i> | 687 |