1

Foundations

1.1 Rudiments on normed algebras

Introduction In this section we develop the basic theory of normed algebras, putting special emphasis on the case of complete normed unital associative complex algebras. Non-associative normed algebras are considered only when they do not offer special difficulties, or when the difficulties can be overcome in an elementary way. Thus, this section is mainly devoted to attracting the attention of the non-expert reader, although the expert reader should browse through it in order to become familiar with the definitions and symbols introduced here, which need to be kept in mind throughout the whole book.

Subsection 1.1.1 deals with the basic spectral theory, and culminates with the proof in Theorem 1.1.46 of the celebrated Gelfand–Beurling formula. In Subsection 1.1.2, we prove Rickart's dense-range-homomorphism theorem. Subsection 1.1.3 deals with the Gelfand theory for complete normed unital associative and commutative complex algebras, as stated in Theorem 1.1.73, and some applications are discussed. In Subsection 1.1.4, we introduce topological divisors of zero in a normed algebra, and involve this notion to prove in Corollary 1.1.95 that (bounded linear) operators on a Banach space are neither bounded below nor surjective whenever they lie in the boundary of the set of all bijective operators. Subsections 1.1.5 and 1.1.6 discuss the complexification, the unital extension, and the completion of a normed algebra. These tools allow us to show how many results, proved originally for complete normed unital associative complex algebras, remain true (sometimes in a suitably altered form) for general normed associative algebras. This section, and throughout, concludes with a subsection of historical notes and comments.

1.1.1 Basic spectral theory

Throughout this work, $\mathbb{K}$ will stand for the field of real or complex numbers. Given vector spaces X,Y over $\mathbb{K}$, we denote by $L(X,Y)$ the vector space over $\mathbb{K}$ of all linear mappings from X to Y, and we set $L(X) := L(X,X)$.

By an *algebra* over $\mathbb{K}$ we mean a vector space A over $\mathbb{K}$ endowed with a bilinear mapping $(a,b) \to ab$ from $A \times A$ to A, which is called the *product* or the *multiplication* of A. An algebra is said to be *associative* (respectively, *commutative*) if its product is associative (respectively, commutative). An element e of an algebra A is said to be a *unit* for A if $ea = ae = a$ for every $a \in A$. Clearly, an algebra has at most a unit. The algebra A is said to be *unital* if it has a nonzero unit (equivalently, if A has a unit and $A \neq 0$). The unit of a given unital algebra will be denoted by $\mathbf{1}$ unless otherwise stated. Given subsets B, C of an algebra A, we set

$$BC := \{xy : (x,y) \in B \times C\}.$$

Exceptionally, but never in Chapter 1, we will consider algebras over an arbitrary field $\mathbb{F}$. These are defined as above with $\mathbb{F}$ instead of $\mathbb{K}$.

Example 1.1.1 (a) Let E be a non-empty set. Then the set $F^{\mathbb{K}}(E)$ (of all functions from E to $\mathbb{K}$), with operations defined pointwise, becomes a unital associative and commutative algebra over $\mathbb{K}$.

(b) Let X be a nonzero vector space over $\mathbb{K}$. Then the vector space $L(X)$, with the product defined as the composition of mappings, becomes a unital associative algebra over $\mathbb{K}$. It is easily realized that $L(X)$ is commutative if and only if $\dim(X) = 1$. We denote by I_X the unit of $L(X)$, namely the identity mapping on X.

(c) By a *subalgebra* of an algebra A over $\mathbb{K}$ we mean a (vector) subspace (say B) of A such that $BB \subseteq B$. In this way, subalgebras of algebras become new examples of algebras.

Let A and B be algebras over $\mathbb{K}$. By an *algebra homomorphism* from A to B we mean a linear mapping $F : A \to B$ satisfying $F(xy) = F(x)F(y)$ for all $x, y \in A$. We say that A and B are *(algebra) isomorphic* if there exists a bijective algebra homomorphism from A to B.

Exercise 1.1.2 Prove that every one-dimensional algebra over $\mathbb{K}$ with nonzero product is isomorphic to $\mathbb{K}$.

A norm $\|\cdot\|$ on (the vector space of) an algebra A over $\mathbb{K}$ is said to be an *algebra norm* if the inequality $\|ab\| \leqslant \|a\|\|b\|$ holds for all $a, b \in A$. By a *normed algebra* we mean an algebra A over $\mathbb{K}$ endowed with an algebra norm. A normed algebra A is said to be *complete* if it becomes a complete metric space under the distance $d(a,b) := \|a - b\|$, i.e. if the normed space underlying A is a Banach space.

§1.1.3 It is clear that the product of any normed algebra is continuous. Actually, the axiom $\|ab\| \leqslant \|a\|\|b\|$ of normed algebras does not give much more. Indeed, if $|||\cdot|||$ is a norm on an algebra A over $\mathbb{K}$ making the product of A continuous (say $|||ab||| \leqslant M|||a||| |||b|||$ for all $a, b \in A$ and some positive number M), then by setting $\|\cdot\| := M|||\cdot|||$, we are provided with an equivalent norm on A converting A into a normed algebra.

Given a normed space X over $\mathbb{K}$, we denote by

$$\mathbb{B}_X := \{x \in X : \|x\| \leqslant 1\}$$

the closed unit ball of X, by

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\}$$

the unit sphere of X, and by X' the (topological) dual of X. When necessary, every normed space X will be seen as a subspace of its bidual X''. Given normed spaces

X,Y over $\mathbb{K}$, we denote by $BL(X,Y)$ the normed space over $\mathbb{K}$ of all bounded linear mappings from X to Y, and we set $BL(X) := BL(X,X)$.

Example 1.1.4 (a) Let E be a locally compact Hausdorff topological space. Then the subalgebra $C_0^{\mathbb{K}}(E)$ of $F^{\mathbb{K}}(E)$ (consisting of all $\mathbb{K}$-valued continuous functions on E vanishing at infinity), endowed with the norm

$$\|x\| := \max\{|x(t)| : t \in E\},$$

becomes a complete normed associative and commutative algebra over $\mathbb{K}$. This algebra is unital if and only if E is compact. When this is the case, we also write $C^{\mathbb{K}}(E)$ instead of $C_0^{\mathbb{K}}(E)$. We note that, by taking E equal to $\mathbb{N}$ endowed with the discrete topology, we obtain that the real or complex Banach space c_0 (of all null sequences in $\mathbb{K}$) naturally becomes a complete normed associative and commutative algebra over $\mathbb{K}$.

(b) Let X be a nonzero normed space over $\mathbb{K}$. Then $BL(X)$ is a subalgebra of $L(X)$, and, endowed with the operator norm

$$\|F\| := \sup\{\|F(x)\| : x \in \mathbb{B}_X\},$$

becomes a normed unital associative algebra over $\mathbb{K}$. Moreover, the normed algebra $BL(X)$ is complete if and only if X is a Banach space. Involving the Hahn–Banach theorem, it is easily realized that $BL(X)$ is commutative if and only if $\dim(X) = 1$.

(c) By restricting the norm, any subalgebra of a normed algebra will be seen without notice as a new normed algebra.

(d) Let E be a topological space, and let A be a normed algebra over $\mathbb{K}$. Then, by the continuity of the product of A, the vector space $C(E,A)$ of all continuous functions from E to A becomes an algebra over $\mathbb{K}$ under the product defined pointwise. The subalgebra $C_b(E,A)$ of $C(E,A)$, consisting of all bounded continuous functions from E to A, becomes a normed algebra over $\mathbb{K}$ under the sup norm.

§1.1.5 An element e of an algebra is said to be an *idempotent* if $e^2 = e$. If e is a nonzero idempotent in a normed algebra, then we clearly have $\|e\| \geqslant 1$. In particular, the unit $\mathbf{1}$ of a normed unital algebra satisfies $\|\mathbf{1}\| \geqslant 1$. Moreover, no more can be said. Indeed, if M is any real number with $M \geqslant 1$, and if for $\lambda \in \mathbb{K}$ we set $\|\lambda\| := M|\lambda|$, where $|\cdot|$ stands for the usual module on $\mathbb{K}$, then $\|\cdot\|$ becomes an algebra norm on $\mathbb{K}$ satisfying $\|1\| = M$.

Fact 1.1.6 *Let E be a connected topological space, let A be a normed algebra over $\mathbb{K}$, let t_0 be in E, and let B stand for the subalgebra of $C(E,A)$ consisting of those continuous functions from E to A vanishing at t_0. Then B has no nonzero idempotent.*

Proof Assume to the contrary that there is a nonzero idempotent $e \in B$. Then $e(t)$ is an idempotent in A for every $t \in E$, and $e(t)$ is nonzero for some $t \in E$. Therefore, since $e(t_0) = 0$, the continuous mapping $t \to \|e(t)\|$ from E to $\mathbb{R}$ would have a disconnected range (cf. §1.1.5 above), contradicting the connectedness of E. □

As an application of §1.1.3, we have the following.

Proposition 1.1.7 *Let A be a finite-dimensional algebra over $\mathbb{K}$. Then A can be provided with an algebra norm.*

Proof Let $\{u_1,\dots,u_n\}$ be a basis of A. Then, for $j,k=1,\dots,n$, we have $u_ju_k = \sum_{i=1}^n \rho_i^{jk}u_i$, for suitable $\rho_1^{jk},\dots,\rho_n^{jk}\in\mathbb{K}$. Now, for $a=\sum_{i=1}^n \lambda_i u_i$, set $|||a||| := \sum_{i=1}^n |\lambda_i|$. Then, for a as above and $b=\sum_{i=1}^n \mu_i u_i$, we have

$$ab = \sum_{i=1}^n \tau_i u_i, \quad \text{with} \quad \tau_i := \sum_{j,k=1}^n \lambda_j\mu_k\rho_i^{jk},$$

and hence

$$|||ab||| = \sum_{i=1}^n |\tau_i| \leqslant M \sum_{j,k=1}^n |\lambda_j||\mu_k| = M|||a|||\,|||b|||,$$

where $M := n\max\{|\rho_i^{jk}| : i,j,k=1,\dots,n\}$ does not depend on the couple (a,b). Finally, apply §1.1.3. □

Lemma 1.1.8 *Let X be a Banach space over $\mathbb{K}$, let Y,Z be normed spaces over $\mathbb{K}$, and let $f : X\times Y\to Z$ be a separately continuous bilinear mapping. Then f is (jointly) continuous.*

Proof For $u\in X$ (respectively, $v\in Y$), let us denote by f_u (respectively, f_v) the bounded linear mapping from Y to Z (respectively, from X to Z) defined by $f_u(y) := f(u,y)$ for every $y\in Y$ (respectively, $f_v(x) := f(x,v)$ for every $x\in X$). Then $\mathscr{F} := \{f_v : v\in\mathbb{B}_Y\}$ is a pointwise bounded family of bounded linear mappings from the Banach space X to the normed space Z. Indeed, for each $x\in X$ and every $v\in\mathbb{B}_Y$ we have

$$\|f_v(x)\| = \|f(x,v)\| = \|f_x(v)\| \leqslant \|f_x\|.$$

It follows from the uniform boundedness principle that $\mathscr{F}$ is uniformly bounded on $\mathbb{B}_X$. This implies the existence of a positive number M satisfying

$$\|f(x,y)\| \leqslant M\|x\|\|y\| \text{ for every } (x,y)\in X\times Y.$$ □

By combining §1.1.3 and Lemma 1.1.8, we obtain the following.

Proposition 1.1.9 *Let A be an algebra over $\mathbb{K}$ endowed with a complete norm $\|\cdot\|$ making the product of A separately continuous. Then, up to the multiplication of $\|\cdot\|$ by a suitable positive number, A becomes a complete normed algebra.*

Definition 1.1.10 Let A be an algebra over $\mathbb{K}$. The *annihilator*, $\mathrm{Ann}(A)$, of A is defined by

$$\mathrm{Ann}(A) := \{a\in A : aA = Aa = 0\}.$$

By a *centralizer* on A we mean a linear mapping (say f) from A to A satisfying $f(ab) = f(a)b = af(b)$ for all $a,b\in A$. The set Γ_A of all centralizers on A is a subalgebra of $L(A)$ containing I_A. This subalgebra is called the *centroid* of A. The algebra A is said to be *central* over $\mathbb{K}$ whenever $\Gamma_A = \mathbb{K}I_A$.

The next proposition contains an easy ‘automatic continuity theorem’.

Proposition 1.1.11 *Let A be an algebra over $\mathbb{K}$ with $\mathrm{Ann}(A)=0$. We have:*

(i) *Γ_A is a commutative algebra.*
(ii) *If A is complete normed, then $\Gamma_A\subseteq BL(A)$.*

Proof Given $f,g \in \Gamma_A$ and $a,b \in A$, we see that

$$\begin{aligned}(f\circ g)(a)b &= f(g(a))b = g(a)f(b) = g(af(b))\\ &= g(f(a)b) = g(f(a))b = (g\circ f)(a)b\end{aligned}$$

and

$$\begin{aligned}b(f\circ g)(a) &= bf(g(a)) = f(b)g(a) = g(f(b)a)\\ &= g(bf(a)) = bg(f(a)) = b(g\circ f)(a).\end{aligned}$$

It follows from the arbitrariness of b in A that $(f\circ g - g\circ f)(a)$ belongs to $\mathrm{Ann}(A)$. Therefore $(f\circ g - g\circ f)(a) = 0$. Now, since a is arbitrary in A, we conclude that $f\circ g = g\circ f$. Thus Γ_A is a commutative algebra.

Assume that A is complete normed. Let f be in Γ_A, and let a_n be a sequence in A with $a_n \to 0$ and $f(a_n) \to b \in A$. Then, for every $a \in A$ we have

$$0 \leftarrow f(a)a_n = af(a_n) \to ab \quad\text{and}\quad 0 \leftarrow a_n f(a) = f(a_n)a \to ba,$$

hence $ab = ba = 0$. Since a is arbitrary in A, and A has zero annihilator, we get that $b = 0$. Thus the continuity of f follows from the closed graph theorem. $\square$

Let A be a unital associative algebra. An element $x \in A$ is said to be *invertible* in A if there exists $y \in A$ such that $xy = yx = \mathbf{1}$. If x is invertible, then the element y above is unique, is called the *inverse* of x, and is denoted by x^{-1}. We denote by $\mathrm{Inv}(A)$ the set of all invertible elements of A.

Example 1.1.12 (a) Let E be a non-empty set, let $F^{\mathbb{K}}(E)$ be as in Example 1.1.1(a), and let x be in $F^{\mathbb{K}}(E)$. Then $x \in \mathrm{Inv}(F^{\mathbb{K}}(E))$ if and only if $x(t) \neq 0$ for every $t \in E$.

(b) Let X be a nonzero vector space over $\mathbb{K}$, and let F be in $L(X)$. Then $F \in \mathrm{Inv}(L(X))$ if and only if F is bijective.

(c) Let E be a compact Hausdorff topological space, let $C^{\mathbb{K}}(E)$ be as in Example 1.1.4(a), and let x be in $C^{\mathbb{K}}(E)$. Then $x \in \mathrm{Inv}(C^{\mathbb{K}}(E))$ if and only if $x(t) \neq 0$ for every $t \in E$. Therefore $x \in \mathrm{Inv}(C^{\mathbb{K}}(E))$ if and only if $x \in \mathrm{Inv}(F^{\mathbb{K}}(E))$.

(d) Let X be a nonzero normed space over $\mathbb{K}$, and let F be in $BL(X)$. Then $F \in \mathrm{Inv}(BL(X))$ if and only if F is bijective and F^{-1} is continuous. Therefore, in the case that X is in fact a Banach space, the Banach isomorphism theorem gives that $F \in \mathrm{Inv}(BL(X))$ if and only if F is bijective, and hence $F \in \mathrm{Inv}(BL(X))$ if and only if $F \in \mathrm{Inv}(L(X))$.

Lemma 1.1.13 *Let A be a normed unital associative algebra over $\mathbb{K}$, and let a and b be in $\mathrm{Inv}(A)$. Then we have:*

(i) $\|a^{-1} - b^{-1}\| \leqslant \|a^{-1}\|\,\|b^{-1}\|\,\|a-b\|$.
(ii) $\left|\frac{1}{\|a^{-1}\|} - \frac{1}{\|b^{-1}\|}\right| \leqslant \|a-b\|$.
(iii) *If* $\|a-b\| < \frac{1}{\|a^{-1}\|}$, *then* $\|b^{-1}\| \leqslant \frac{\|a^{-1}\|}{1-\|a^{-1}\|\|a-b\|}$.

Proof We have

$$\|a^{-1} - b^{-1}\| = \|a^{-1}(b-a)b^{-1}\| \leqslant \|a^{-1}\|\,\|b^{-1}\|\,\|a-b\|,$$

which proves assertion (i). Now, keeping in mind that

$$\|b^{-1}\| - \|a^{-1}\| \leqslant \|a^{-1} - b^{-1}\|,$$

it follows from assertion (i) that

$$\frac{1}{\|a^{-1}\|}-\frac{1}{\|b^{-1}\|}\leqslant\|a-b\|. \tag{1.1.1}$$

The proof of assertion (ii) is concluded by combining the inequality (1.1.1) with the one obtained by interchanging the roles of a and b. On the other hand, it follows from the inequality (1.1.1) that

$$\frac{1-\|a^{-1}\|\|a-b\|}{\|a^{-1}\|}=\frac{1}{\|a^{-1}\|}-\|a-b\|\leqslant\frac{1}{\|b^{-1}\|}. \tag{1.1.2}$$

Since the condition $\|a-b\|<\frac{1}{\|a^{-1}\|}$ leads to $1-\|a^{-1}\|\|a-b\|>0$, assertion (iii) follows from (1.1.2). □

Corollary 1.1.14 *Let A be a normed unital associative algebra over $\mathbb{K}$, let a be in A, and let z be in $\mathbb{K}$ such that $a-z\mathbf{1}\in\mathrm{Inv}(A)$ and $|z|>\|\mathbf{1}\|\,\|a\|$. Then*

$$\|(a-z\mathbf{1})^{-1}\|\leqslant\frac{\|\mathbf{1}\|}{|z|-\|\mathbf{1}\|\,\|a\|}.$$

Proof We have $\|-z\mathbf{1}-(a-z\mathbf{1})\|=\|a\|<\dfrac{1}{\|(z\mathbf{1})^{-1}\|}$, and hence, by Lemma 1.1.13(iii),

$$\|(a-z\mathbf{1})^{-1}\|\leqslant\frac{\|(z\mathbf{1})^{-1}\|}{1-\|(z\mathbf{1})^{-1}\|\|a\|}=\frac{\|\mathbf{1}\|}{|z|-\|\mathbf{1}\|\,\|a\|}. \qquad \square$$

Let A be a unital associative algebra over $\mathbb{K}$. It is straightforward that ab and a^{-1} belong to $\mathrm{Inv}(A)$ whenever a,b are in $\mathrm{Inv}(A)$. As a consequence, the set $\mathrm{Inv}(A)$ is a group with respect to the product of A. We recall that a *topological group* is a group G endowed with a topology making the mappings $(x,y)\to xy$ from $G\times G$ to G, and $x\to x^{-1}$ from G to G, continuous.

Proposition 1.1.15 *Let A be a normed unital associative algebra over $\mathbb{K}$. Then $\mathrm{Inv}(A)$ is a topological group in the induced topology from A.*

Proof Keeping in mind the continuity of the product of A, it only remains to verify the continuity of the mapping $x\to x^{-1}$ from $\mathrm{Inv}(A)$ to A. By Lemma 1.1.13(ii), the mapping $x\to\frac{1}{\|x^{-1}\|}$ from $\mathrm{Inv}(A)$ to $\mathbb{R}$ is continuous, and hence so is the mapping $x\to\|x^{-1}\|$. Now, the proof concludes by invoking Lemma 1.1.13(i). □

§**1.1.16** Let A be a normed associative algebra over $\mathbb{K}$, and let a be in A. We define powers of a by $a^1:=a$ and $a^{n+1}:=aa^n$. It is easily realized that $a^{n+m}=a^na^m$ for all $n,m\in\mathbb{N}$. Now assume that A is normed. Then we define the *spectral radius* $\mathfrak{r}(a)$ of a by

$$\mathfrak{r}(a):=\inf\left\{\|a^n\|^{\frac{1}{n}}:n\in\mathbb{N}\right\}.$$

Obviously, $\mathfrak{r}(a)\leqslant\|a\|$ and $\mathfrak{r}(\lambda a)=|\lambda|\mathfrak{r}(a)$ for $\lambda\in\mathbb{K}$. It is also clear that, as the infimum of a family of continuous functions, $\mathfrak{r}(\cdot)$ becomes an upper semicontinuous function on A.

Lemma 1.1.17 *Let α_n be a sequence of non-negative real numbers satisfying*

$$\alpha_{n+m} \leqslant \alpha_n \alpha_m \text{ for all } n,m \in \mathbb{N}.$$

Then the limit $\lim \alpha_n^{\frac{1}{n}}$ *exists and is equal to* $\inf\{\alpha_n^{\frac{1}{n}} : n \in \mathbb{N}\}$.

Proof Write $\alpha = \inf\{\alpha_n^{\frac{1}{n}} : n \in \mathbb{N}\}$ and let $\varepsilon > 0$. Fix k such that $\alpha_k^{\frac{1}{k}} < \alpha + \varepsilon$. Any natural number $n \geqslant k$ can be written uniquely in the form $n = q(n)k + r(n)$, where $q(n) \in \mathbb{N}$ and $0 \leqslant r(n) \leqslant k-1$, and hence, setting α_0 equal to 1, we obtain

$$\alpha_n \leqslant \alpha_{r(n)} \alpha_k^{q(n)} \leqslant \max\{1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}\} (\alpha + \varepsilon)^{q(n)k}.$$

Since $\frac{r(n)}{n} \to 0$, we have $\frac{q(n)k}{n} \to 1$ as $n \to \infty$, and hence

$$\alpha_n^{\frac{1}{n}} \leqslant \max\{1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}^{\frac{1}{n}} (\alpha + \varepsilon)^{\frac{q(n)k}{n}} \to \alpha + \varepsilon$$

as $n \to \infty$. Thus $\limsup \alpha_n^{\frac{1}{n}} \leqslant \alpha + \varepsilon$ and, since ε was arbitrary and $\alpha \leqslant \alpha_n^{\frac{1}{n}}$ for every n, we conclude that $\lim \alpha_n^{\frac{1}{n}} = \alpha$. □

Corollary 1.1.18 *Let A be a normed associative algebra over $\mathbb{K}$, and let a be in A. We have:*

(i) $\mathfrak{r}(a) = \lim \|a^n\|^{\frac{1}{n}}$.
(ii) *If $\mathfrak{r}(a) < 1$, then the sequence a^n converges to zero.*

Proof Assertion (i) follows from Lemma 1.1.17 above and the fact that

$$\|a^{n+m}\| \leqslant \|a^n\| \, \|a^m\| \quad \text{for all} \quad n,m \in \mathbb{N}.$$

Assume that $\mathfrak{r}(a) < 1$. Choose $\mathfrak{r}(a) < \eta < 1$. By assertion (i), we have $\|a^n\|^{\frac{1}{n}} < \eta$ for $n \in \mathbb{N}$ large enough, and hence $\|a^n\| < \eta^n \to 0$. Thus assertion (ii) has been proved. □

Corollary 1.1.19 *Let A and B be normed associative algebras over $\mathbb{K}$, let $F : A \to B$ be a continuous algebra homomorphism, and let a be in A. Then $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$. As a consequence, every equivalent algebra norm on A gives rise to the same spectral radius on A.*

Proof For $n \in \mathbb{N}$, we have

$$\|F(a)^n\| = \|F(a^n)\| \leqslant \|F\| \|a^n\|.$$

Therefore, by taking nth roots, and letting $n \to \infty$, Corollary 1.1.18(i) gives $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$. □

Lemma 1.1.20 (von Neumann) *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let a be in A with $\mathfrak{r}(a) < 1$. Then $\mathbf{1} - a \in \mathrm{Inv}(A)$ and*

$$(\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n,$$

where $a^0 := \mathbf{1}$.

Proof Choose η with $\mathfrak{r}(a) < \eta < 1$. By Corollary 1.1.18(i), we have $\|a^n\| \leqslant \eta^n$ for n large enough, and therefore the series $\sum \|a^n\|$ converges. It follows from the completeness of A that the series $\sum a^n$ is convergent in A. Since for each n we have

$$(\mathbf{1}-a)(\mathbf{1}+a+\cdots+a^n) = (\mathbf{1}+a+\cdots+a^n)(\mathbf{1}-a) = \mathbf{1}-a^{n+1},$$

it follows that

$$(\mathbf{1}-a)\left(\sum_{n=0}^{\infty} a^n\right) = \left(\sum_{n=0}^{\infty} a^n\right)(\mathbf{1}-a) = \mathbf{1}.$$

Thus $\mathbf{1}-a$ is an invertible element of A, and its inverse is $\sum_{n=0}^{\infty} a^n$. □

Corollary 1.1.21 *Let A be a complete normed unital associative algebra over $\mathbb{K}$. We have:*

(i) *If $a \in A$ satisfies $\|\mathbf{1}-a\| < 1$, then $a \in \mathrm{Inv}(A)$.*
(ii) *If $a \in \mathrm{Inv}(A)$, and if $b \in A$ satisfies $\|a-b\| < \frac{1}{\|a^{-1}\|}$, then $b \in \mathrm{Inv}(A)$.*

Proof Assertion (i) follows by writing $a = \mathbf{1} - (\mathbf{1}-a)$ and by applying Lemma 1.1.20. Given $a \in \mathrm{Inv}(A)$ and $b \in A$ such that $\|a-b\| < \frac{1}{\|a^{-1}\|}$, we see that

$$\|\mathbf{1}-a^{-1}b\| = \|a^{-1}(a-b)\| \leqslant \|a^{-1}\|\|a-b\| < 1.$$

Therefore, by assertion (i), $a^{-1}b \in \mathrm{Inv}(A)$, and so $b = a(a^{-1}b) \in \mathrm{Inv}(A)$. □

Lemma 1.1.22 *Let A be a unital associative algebra over $\mathbb{K}$, and let x and y be in* $\mathrm{Inv}(A)$. *Then*

$$x^{-1}-y^{-1}-y^{-1}(y-x)y^{-1} = y^{-1}(y-x)x^{-1}(y-x)y^{-1}.$$

Proof We have

$$\begin{aligned} x^{-1}-y^{-1}-y^{-1}(y-x)y^{-1} &= x^{-1}(y-x)y^{-1}-y^{-1}(y-x)y^{-1} \\ &= (x^{-1}-y^{-1})(y-x)y^{-1} \\ &= y^{-1}(y-x)x^{-1}(y-x)y^{-1}. \end{aligned}$$ □

Let X,Y be normed spaces over $\mathbb{K}$, let Ω be a non-empty open subset of X, let x_0 be in Ω, and let $f:\Omega \to Y$ be a function. We recall that f is said to be *(Fréchet) differentiable at* x_0 if there exists $T \in BL(X,Y)$ such that

$$\lim_{\substack{x\to x_0 \\ x\in\Omega\setminus\{x_0\}}} \frac{\|f(x)-f(x_0)-T(x-x_0)\|}{\|x-x_0\|} = 0.$$

In this case, the operator T is unique, and is called the *(Fréchet) derivative of f at* x_0. When $X=\mathbb{K}$, the natural identification $BL(\mathbb{K},Y) \equiv Y$ allows us to see the derivative of f at x_0 as the element $f'(x_0) \in Y$ given by

$$f'(x_0) := \lim_{\substack{x\to x_0 \\ x\in\Omega\setminus\{x_0\}}} \frac{f(x)-f(x_0)}{x-x_0}.$$

Theorem 1.1.23 *Let A be a complete normed unital associative algebra over $\mathbb{K}$. Then* $\mathrm{Inv}(A)$ *is open in A. Moreover, the mapping $x \to x^{-1}$ from* $\mathrm{Inv}(A)$ *to A is differentiable at any point $a \in \mathrm{Inv}(A)$, with derivative equal to the mapping $x \to -a^{-1}xa^{-1}$ from A to A.*

Proof The first conclusion follows from Corollary 1.1.21(ii). Let us fix $a \in \mathrm{Inv}(A)$. Then, by Lemma 1.1.22, for each $x \in \mathrm{Inv}(A)$ we have

$$x^{-1} - a^{-1} - [-a^{-1}(x-a)a^{-1}] = a^{-1}(x-a)x^{-1}(x-a)a^{-1},$$

and hence

$$\|x^{-1} - a^{-1} - [-a^{-1}(x-a)a^{-1}]\| \leqslant \|a^{-1}\|^2\,\|x^{-1}\|\,\|x-a\|^2.$$

Since the mapping $x \to \|x^{-1}\|$ is continuous (by Lemma 1.1.13(ii)), we derive

$$\lim_{\substack{x \to a \\ x \in \mathrm{Inv}(A)\setminus\{a\}}} \frac{\|x^{-1} - a^{-1} - [-a^{-1}(x-a)a^{-1}]\|}{\|x-a\|} = 0.$$

Therefore, the mapping $x \to x^{-1}$ is differentiable at a with derivative the mapping $T \in BL(A)$ given by $T(x) = -a^{-1}xa^{-1}$. □

§**1.1.24** Let A be an algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Since the intersection of any family of subalgebras of A is again a subalgebra of A, it follows that the intersection of all subalgebras of A containing S is the smallest subalgebra of A containing S. This subalgebra is called *the subalgebra of A generated by S*, and is denoted by $A(S)$.

Exercise 1.1.25 Let A be a unital algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Prove that $A(S \cup \{\mathbf{1}\}) = \mathbb{K}\mathbf{1} + A(S)$.

Now, let A be a normed algebra, and let S be a non-empty subset of A. Since the intersection of any family of closed subalgebras of A is again a closed subalgebra of A, it follows that the intersection of all closed subalgebras of A containing S is the smallest closed subalgebra of A containing S. This subalgebra is called *the closed subalgebra of A generated by S*, and is denoted by $\bar{A}(S)$.

Exercise 1.1.26 Let A be a normed algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Prove that:

(i) If S is a subalgebra of A, then so is $\bar{S}$.
(ii) $\bar{A}(S) = \overline{A(S)}$.
(iii) If A is unital, then $\bar{A}(S \cup \{\mathbf{1}\}) = \mathbb{K}\mathbf{1} + \bar{A}(S)$.

§**1.1.27** As usual, we denote by $\mathbb{K}[\mathbf{x}]$ the algebra of all polynomials in the indeterminate $\mathbf{x}$ with coefficients in $\mathbb{K}$. Let A be a unital associative algebra over $\mathbb{K}$, and let $a \in A$. Given a polynomial $p(\mathbf{x}) = \sum_{k=0}^{n} \alpha_k \mathbf{x}^k$ with coefficients $\alpha_k \in \mathbb{K}$, we denote by $p(a)$ the element of A given by $p(a) = \sum_{k=0}^{n} \alpha_k a^k$. It is clear that the mapping $p \to p(a)$ is a unit-preserving algebra homomorphism from $\mathbb{K}[\mathbf{x}]$ onto the subalgebra of A generated by $\mathbf{1}$ and a.

If A is a unital associative algebra and if B is a subalgebra of A containing the unit of A, then it is clear that $\mathrm{Inv}(B) \subseteq B \cap \mathrm{Inv}(A)$. The next example shows that the reverse inclusion may not, in general, be true, even in a complete normed context.

Example 1.1.28 Consider the complete normed unital associative and commutative algebra $C^{\mathbb{C}}(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let u be the element of $C^{\mathbb{C}}(\mathbb{T})$ given by $u(z) = z$ for every $z \in \mathbb{T}$. It is clear that u is invertible in $C^{\mathbb{C}}(\mathbb{T})$ and that the inverse of u is the function defined by $u^{-1}(z) = \frac{1}{z}$ for every $z \in \mathbb{T}$. Let B (respectively, C) denote the subalgebra (respectively, closed subalgebra) of $C^{\mathbb{C}}(\mathbb{T})$ generated by $\{\mathbf{1}, u\}$. Note that B is nothing other than the subalgebra of $C^{\mathbb{C}}(\mathbb{T})$ consisting of all complex polynomial functions, and that $C = \bar{B}$ because of Exercise 1.1.26(ii). If u were invertible in C, then we would have $u^{-1} \in C$, and therefore there would be a polynomial function p satisfying $\|u^{-1} - p\| < 1$. Thus, for $z \in \mathbb{T}$ we would have $|\frac{1}{z} - p(z)| < 1$, and hence $|1 - zp(z)| < 1$. Then, by the maximum modulus principle, the inequality $|1 - zp(z)| < 1$ would be true for every $z \in \mathbb{B}_{\mathbb{C}}$, and in particular $1 = |1 - 0p(0)| < 1$. This contradiction shows that u is not invertible in C.

§**1.1.29** Given an element a in a complete normed unital associative algebra A, $\exp(a)$ is defined as the element of A given by

$$\exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

where $a^0 := \mathbf{1}$.

Exercise 1.1.30 Let a and b be commuting elements of a complete normed unital associative algebra A. Prove that

$$\exp(a+b) = \exp(a)\exp(b), \quad \exp(a) \in \mathrm{Inv}(A), \quad \text{and} \quad \exp(a)^{-1} = \exp(-a).$$

Let A be a unital associative algebra over $\mathbb{K}$. By a *one-parameter semigroup* in A we mean a mapping $S : \mathbb{R}_0^+ \to A$ satisfying

(i) $S(0) = \mathbf{1}$.
(ii) $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$.

If A is complete normed, and if a is any element of A, then it is clear that the mapping $S : \mathbb{R}_0^+ \to A$ defined by $S(t) := \exp(ta)$ becomes a continuous one-parameter semigroup in A. Conversely, we have the following.

Theorem 1.1.31 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let $S : \mathbb{R}_0^+ \to A$ be a continuous one-parameter semigroup in A. Then there exists an element a in A such that $S(t) = \exp(ta)$ for every $t \in \mathbb{R}_0^+$. Moreover, this element is given by the formula*

$$a = \lim_{t \to 0} \frac{S(t) - \mathbf{1}}{t}.$$

Proof Since S is continuous, the integral $\int_\alpha^\beta S(t)\,dt$ exists for all $\alpha, \beta \in \mathbb{R}_0^+$ and is an element of A. Further, by the fundamental theorem of calculus, we have

$$\lim_{\beta \to \alpha} \frac{1}{\beta - \alpha} \int_\alpha^\beta S(t)\,dt = S(\alpha)$$