Introduction

The theory of one-dimensional complex dynamical systems, understood as the global study of iteration of holomorphic mappings, has its roots in the early twentieth century with the work of Pierre Fatou and Gaston Julia. Local studies had been successfully attempted earlier, but it was Fatou and Julia's seminal work, inspired by Paul Montel's notion of normal families of mappings (then relatively new), that set the basis of what is known today as *holomorphic dynamics*. Both Fatou and Julia studied extensively the basic partition of the dynamical space into the two disjoint, completely invariant subsets, *the Fatou set*, which is the open set where tame dynamics occur – the set where Montel's normality appears – and its complement, *the Julia set*, which is the set of initial values whose orbits are chaotic.

Their greatest achievement, arguably, is their detailed description of the geometry and the dynamics of the connected components of the Fatou set, called the 'Fatou components'. Their *Classification Theorem* asserts that every periodic Fatou component of a holomorphic map of the Riemann sphere (a rational map) is either (i) a component of an immediate basin of attraction of some attracting or parabolic cycle or (ii) a rotation domain conformally equivalent to a disc or an annulus.

Their work also left many interesting open questions, such as Fatou's *No Wandering Domains Conjecture*, which states that all Fatou components are eventually periodic, and waited some 60 years for its resolution. Some of their other questions remain open today, but despite their compelling interest, the problems were largely ignored until the late 1970s.

The other side of our story, the theory of quasiconformal functions of the plane, began at roughly the same time with the work of Herbert Grötzsch. As the name suggests, quasiconformality is a weakening of the notion of conformality. It is best understood as a geometric condition: conformal maps preserve angles, and quasiconformal maps distort angles, *but only in a bounded fashion.* Another important difference is that they do not need to be differentiable everywhere, but only almost everywhere. Even so, many important theorems about conformal mappings, minimally recast, remain true for quasiconformal maps.

Grötzsch's work was soon taken in two different directions. Oswald Teichmüller showed that there is a deep connection between quasiconformality and the function theory of Riemann surfaces, and Charles B. Morrey investigated its relation to the solution of PDEs. Beginning in the 1950s, the geometric side of the theory was developed by Albert Pfluger and Lars Ahlfors. This led to a unification of all of the earlier work into a single, general theory. At the same time, Morrey and Ahlfors, together with Lipman Bers and Bogdan V. Bojarski formulated the celebrated *Measurable Riemann Mapping Theorem* as it is known today – in this book, it is referred to somewhat more concisely as the *Integrability Theorem*. As we shall see, it is the essential tool for *quasiconformal surgery*.

Around 1980, two remarkable developments added to this account. On the one hand, computer graphics made it possible to draw pictures of the beautiful phenomena and fractal structures exhibited by the holomorphic dynamics of even very simple maps. This clearly awoke an interest in the subject. On the other hand, and at a much deeper level, in 1981 Dennis Sullivan realized there was a strong connection between holomorphic iteration and the actions of Kleinian groups, introducing what has become known as *Sullivan's dictionary* between these two subjects. Inspired by Henri Poincaré's original (1883) perturbations of Fuchsian groups into quasi-Fuchsian groups, he injected the modern theory of quasiconformal mappings into complex iteration to solve Fatou's No Wandering Domains Conjecture. With the same arguments, he also gave a new proof of Ahlfors' Finiteness Theorem, a cornerstone in the theory of Kleinian groups.

Sullivan's technique is now referred to as *soft quasiconformal surgery*. It deals with quasiconformal deformations of a given map. However, what we call *cut and paste quasiconformal surgery* is more reminiscent of *topological surgery*, which is used to produce one manifold from another in some 'controlled way'. In holomorphic iteration, quasiconformal surgery is used to obtain holomorphic maps with prescribed dynamics that arise from certain model maps which are locally quasiconformal. Often, these model maps are constructed by cutting and pasting different spaces and maps together, explaining the reference to topological surgery. It is worth noticing that, whereas the uniqueness of analytic continuation implies that holomorphic maps are rather rigid, quasiconformal maps can be pasted together very flexibly without losing quasiconformality.

Sullivan's first crucial work was merely the beginning of the uses to which quasiconformal surgery could be put in holomorphic dynamics. Quasiconformal surgery has been at the heart of many important advances and is now part of the essential background knowledge of any researcher in the field.

All of this brings us to the *raison d'etre* for this book. While the theory of quasiconformal maps is a mature subject for which an excellent literature exists, a comprehensive treatment of its application to dynamical systems has, until now, only been possible by going back to the original papers. Books about quasiconformal maps, like [Ah5, LV, Le] and [AIM], naturally focus on the function theory, so the surgery techniques are of mainly anecdotal interest. Similarly, general texts on holomorphic dynamics, like [Mi1, Bea, CG, St] and [MNTU], are not the right place to discuss quasiconformal surgery, because the prerequisite theory of quasiconformality would sit very heavily in such a general work. Our goal is to present a text where one can learn from the start about this beautiful, highly geometric and powerful technique and how to apply it to holomorphic dynamics.

This book is therefore neither a book about holomorphic dynamics nor about quasiconformal mappings, although it contains introductions to both subjects, together with a brief introduction to Kleinian groups, in order to understand the parallels that often come up. These introductions, however, do not present proofs that can easily be found in other standard texts. Both authors have a stronger background in dynamics than in analysis and this surely introduces a bias in the way these introductions are presented. While no previous knowledge about quasiconformal maps or Kleinian groups is assumed, some familiarity with dynamical systems and especially holomorphic dynamics in one complex variable is useful, in particular when understanding the purpose of the different surgery constructions.

While the foundations of the theory and the general description of surgery techniques occupy a good half of the book, the other half is completely dedicated to a variety of applications. These applications are more or less independent of each other, and so each can be read on its own. Our interest is precisely to present the different types of surgery techniques that are required for attacking the most important results in holomorphic dynamics. The applications are grouped by similarities between the constructions they require. Two of them go beyond the classical realm of quasiconformal surgery and use trans-quasiconformal surgery. Even further, another one deals with holomorphic correspondences, a natural generalization of holomorphic maps.

Some of the sections on applications are written by the authors of the original papers, whom we once again thank.

We should also comment on what this book does not cover. Although quasiconformal maps and surgery itself are intimately related to Teichmüller Theory, this deep area of mathematics is not treated in this book. To learn about it, we strongly recommend the recent volume by John H. Hubbard [Hu2]. In connection to the above, there is also no reference to quadratic differentials, although they play an important role in the theory of holomorphic dynamics.

Actions of Kleinian groups only appear in the book in a sparse way. We give a brief introduction to the subject so that references to the parallelism between deformations of Kleinian groups and surgery in holomorphic dynamics, expressed by Sullivan's dictionary, can be understood.

About the book structure

Chapter 1, *Quasiconformal geometry*, contains the basic definitions and results about quasiconformal mappings used in surgery, and those necessary to make a coherent description. We give a geometrical and an analytical approach, give precise references to all important results, and prove those (even if simple), which we could not find proven in the literature. Nevertheless they may be useful for the reader of this text.

The same comments apply to Chapter 2, *Boundary behaviour of quasiconformal maps: extensions and interpolations*. We study how to extend maps to the boundary of their domains, depending on the regularity of the elements involved. Conversely, we see how to extend boundary maps to their neighbouring domains, depending also on the regularity of the different ingredients. These extensions and interpolations are technical results, which are used repeatedly in each and every surgery. Some of the results are celebrated theorems while others are simple exercises; but all are part of the community folklore. We have made it a point to collect them in this chapter, and to prove or indicate proofs of those that can be shown with elementary methods.

Chapter 3, *Preliminaries on dynamical systems and actions of Kleinian groups*, contains the background in dynamics necessary for understanding the sections that follow and especially their motivation. We include general basic background for discrete dynamical systems, circle maps, holomorphic dynamics, and families of such, as well as dynamics of Kleinian groups. As in the previous chapters, there are basically no proofs, but all results are well referenced.

In Chapter 4, *Introduction to surgery and first occurrences*, we present the earliest applications of surgery to holomorphic dynamics, although not necessarily the simplest. Inspired by quasiconformal deformations of Kleinian groups, the surgeries lead to conformal parametrizations of hyperbolic components of the Mandelbrot set by $\mathbb{D}$, due to Sullivan, Douady and Hubbard, and to the No Wandering Domains Theorem for rational maps, due to Sullivan. We have chosen to keep a historical flavour, so the surgeries appear in a form fairly close to their original presentation.

When applying quasiconformal surgery, there are several paths one may take to accomplish the same result. More precisely, Chapter 5, *General principles of surgery*, is dedicated to establishing three criteria (due to Mitsuhiro Shishikura and Sullivan) giving conditions under which a surgery can be completed to result in a holomorphic map. The First and Second Shishikura Principles are actually corollaries of the third principle, called Sullivan's Straightening Theorem. We present all three since their proofs provide different insights. Sullivan's Straightening Theorem gives a necessary and sufficient condition to decide whether a surgery can be completed. However, these criteria are rarely used in the applications in the book, since certain informations can only be extracted from the specific details of the proof 'from scratch'. Needless to say, the criteria may be useful in many other instances.

Chapters 6 to 9 are entirely dedicated to applications. Sections are pairwise independent, although they all use the common terminology and background of the first part of the book. Some of the applications were presented during the surgery workshop at the Institute of Henri Poincaré in Paris in the fall of 2003 as part of the trimester organized by Adrien Douady. Other applications have been added for completeness. We have tried to show as many different constructions as possible. However, the list is by no means exhaustive.

Chapter 6, *Soft surgeries*, consists of applications where only the complex structure is deformed, so that it is invariant under the given map. The resulting maps are therefore quasiconformally conjugate to the original one. The chapter contains a contribution by Xavier Buff and Christian Henriksen. Chapter 7, *Cut and paste surgery*, is the longest chapter and contains many groups of applications. The common feature in all of them is that maps and complex structures are both changed. It contains contributions by Kevin M. Pilgrim and Tan Lei, and Shaun Bullett, who present a special surgery application of mating a Kleinian group with a quadratic polynomial. Chapter 8, *Cut and paste surgeries with sectors*, contains applications where the deformation of the maps and the complex structures are concentrated in sectors. The chapter contains a contribution by Adam Epstein and Michael Yampolsky.

Chapter 9, *Trans-quasiconformal surgery*, is dedicated to the contributions by Carsten L. Petersen and Peter Haïssinsky. The surgeries require maps, called *David homeomorphisms*, that are not quite quasiconformal, but are 'close enough' in a sense made precise there. Many new applications have appeared using this technique. The two we have included are among the earliest and a good introduction to the more recent work.

On how to read this book

Readers familiar with quasiconformal geometry and analysis may start in Chapter 3, *Preliminaries on dynamical systems and actions of Kleinian groups*, and continue from there. Those familiar with dynamics, holomorphic dynamics in particular, may start in Chapter 1, *Quasiconformal geometry*, read Chapter 2, *Extensions and interpolations*, only diagonally to see what type of results it contains, and then jump directly to Chapter 4, *Introduction to surgery and first occurrences*. Chapter 2 can be used as a reference chapter throughout the book.

In both cases, if the first applications in Chapter 4 are too difficult to start with, we advise to leave them for later and start with those in Chapter 7, *Cut and paste surgery*, maybe after having read the introduction to surgery given in Chapter 4. Chapter 5, *General principles of surgery*, can be read at any point, since the principles are not used often in the constructions, and they may be too abstract for someone not well acquainted with a number of examples.

We hope that experts in both quasiconformal mappings and holomorphic dynamics, and even in quasiconformal surgery, may still find some of the applications interesting and in any case a practical collection to have at hand as a reference. It is clear that such readers may skip Chapters 1 to 5.

The book can be used, and has been used, several times already, for a graduate course in quasiconformal surgery. If so, it is not recommended to cover the book in a sequential way but to follow the recommendations given above. The lecturer may selectively choose some sections in Chapters 6 to 9 and keep the remaining ones as projects for the students who may be asked to read the original papers, fill in details and do the suggested exercises.

1
Quasiconformal geometry

Since its introduction in the early 1980s quasiconformal surgery has become a major tool in the development of the theory of holomorphic dynamics.

The goal of this chapter is to collect the basic definitions and results about quasiconformal mappings used in surgery, and those necessary to make a coherent description. We give a geometrical and analytical approach and give precise references to important results in the vast literature about quasiconformal mappings. In general we only prove those we could not find proven elsewhere, even if simple. However, since the *Integrability Theorems* (Theorems 1.27 and 1.28), also called the *Measurable Riemann Mapping Theorems*, are the cornerstone behind every surgery construction we sketch a proof of those.

Holomorphic maps are very rigid, due to the property of analytic continuation. For this reason it is not possible to paste different holomorphic maps together along a curve to form a new holomorphic map. However, quasiconformal mappings do have this kind of flexibility and can be pasted together to form new quasiconformal mappings. It is this flexibility that produces the basis for surgery constructions where we change mappings and sometimes also the underlying spaces. When the construction is successful the final goal is to end with a holomorphic map, obtained via the Integrability Theorem.

1.1 The linear case: Beltrami coefficients and ellipses

Let $\mathbb{C}_{\mathbb{R}}$ denote the complex plane, viewed as the two-dimensional oriented Euclidean $\mathbb{R}$-vector space with the orthonormal positively oriented standard basis $\{1, i\}$. In $\mathbb{C}_{\mathbb{R}}$ we shall use as coordinates either (x, y) or $(z, \overline{z})$ where $z = x + iy$ and $\overline{z} = x - iy$.

Any $\mathbb{R}$-linear map $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ can be written, using the coordinates $(z, \overline{z})$, in the form

$$L(z) = az + b\overline{z}, \qquad \text{with } a, b, z \in \mathbb{C}.$$

The unit square, spanned by 1 and i, is mapped onto the parallelogram spanned by $a + b$ and $a\mathrm{i} - b\mathrm{i}$ (see Figure 1.1).

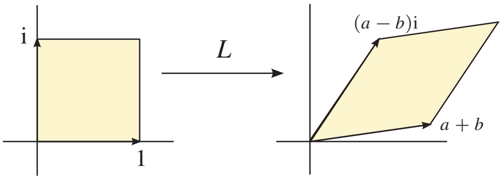

Figure 1.1 The unit square is mapped to the parallelogram spanned by $\{a + b, (a - b)\mathrm{i}\}$.

The absolute value of the determinant of L is the area of the parallelogram, i.e. $\det(L) = |a|^2 - |b|^2$. We shall restrict to $\mathbb{R}$-linear maps that are invertible and orientation preserving, i.e. with $|a| > |b|$.

We define the *Beltrami coefficient* of L to be $\mu(L) = \frac{b}{a}$, and – for reasons which will become clear below – we let $\theta \in \mathbb{R}/(\pi\mathbb{Z})$ denote half the argument of $\mu(L)$, i.e.

$$\mu(L) := \left|\frac{b}{a}\right| e^{\mathrm{i}\,2\theta}.$$

Note that $\mu(L) \in \mathbb{D}$ when L is orientation preserving, and that L is holomorphic if and only if $b = 0$, which occurs if and only if $\mu(L) = 0$.

Let $E(L)$ denote the inverse image by L of the unit circle. Then $E(L)$ is an ellipse, and in particular a circle if $\mu(L) = 0$.

In order to determine the ellipse $E(L)$, we set $a = |a|\ e^{\mathrm{i}\,\alpha}$, where $\alpha \in \mathbb{R}/(2\pi\mathbb{Z})$, $\mu = \mu(L)$ and rewrite L as

$$L(z) = e^{\mathrm{i}\,\alpha}\ |a|\big(z + |\mu|\ e^{\mathrm{i}\,2\theta}\overline{z}\big).$$

Hence, L is the $\mathbb{R}$-linear map $S(z) = |a|\big(z + |\mu|\ e^{\mathrm{i}\,2\theta}\ \overline{z}\big)$ post-composed with the rotation $R(z) = \mathrm{e}^{\mathrm{i}\,\alpha}z$. We have split L into the composition of a self-adjoint linear transformation followed by an orthogonal transformation. (It is easy to check that the 2×2 matrix of S in the basis $\{1, \mathrm{i}\}$ is symmetric, and S therefore is self-adjoint.) It follows that S has two real eigenvalues and, if $b \neq 0$, that their corresponding eigenvector directions are orthogonal (see Figure 1.2).

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More information

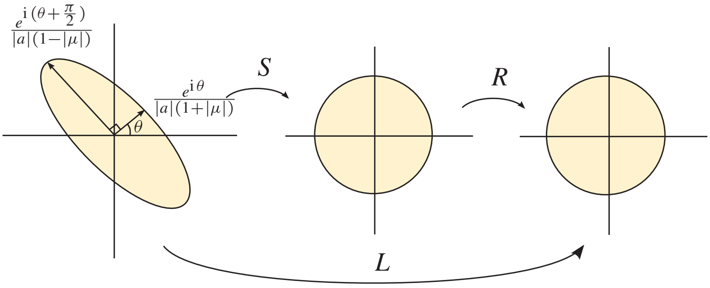

Figure 1.2 The ellipse $E(S) = E(L)$.

It is easy to check that $e^{i\theta}$ and $e^{i(\theta+\pi/2)}$ are eigenvectors of S corresponding to the eigenvalues $|a|(1+|\mu|)$ and $|a|(1-|\mu|)$ respectively. It follows that $E(S)$ is the ellipse with half major axis $\frac{1}{|a|(1-|\mu|)}$ along the direction $e^{i(\theta+\pi/2)}$ and half minor axis $\frac{1}{|a|(1+|\mu|)}$ along the orthogonal direction $e^{i\theta}$. The ellipse $E(L)$ equals $E(S)$, since the unit circle is preserved by the rotation $R(z) = e^{i\alpha}z$.

We define the *dilation* $K(L)$ of L as the ratio of the major axis to the minor axis:

$$K(L) := \frac{1+|\mu|}{1-|\mu|} = \frac{|a|+|b|}{|a|-|b|},$$

and the *complex dilatation* of L as the Beltrami coefficient $\mu(L)$. The dilatation $K(L)$ determines the shape of the ellipse up to scaling, but not the position of its axes. The Beltrami coefficient determines the position and the shape up to scaling. Conversely, if we happen to start with an ellipse E, the Beltrami coefficient is determined by $\mu(E) = \frac{M-m}{M+m}e^{i2\theta}$, where M and m are the half major and half minor axes of E respectively, and θ is the argument of the direction of the minor axis of E chosen in $[0, \pi)$. The dilatation $K(E)$ is defined in the natural way (see Figure 1.3).

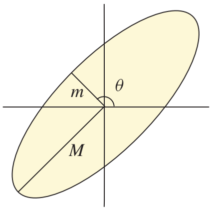

Figure 1.3 Given an ellipse E, we define $\mu(E)$ and $K(E)$ in terms of M, m, and θ. Observe that we choose the argument of the minor axis, θ, to belong to $[0, \pi)$.

We shall denote by σ_0 the standard conformal structure of $\mathbb{C}_{\mathbb{R}}$, that is to consider $\mathbb{C}_{\mathbb{R}}$ as a $\mathbb{C}$-vector space with the standard complex scalar multiplication. Any invertible $\mathbb{R}$-linear map L can be used to define a new conformal structure $\sigma(L)$ on the domain of L, that is a new operation making $\mathbb{C}_{\mathbb{R}}$ into a $\mathbb{C}$-vector space, extending the $\mathbb{R}$-vector space structure. This is done in the following way: we need to define what it means to 'multiply' elements of $\mathbb{C}_{\mathbb{R}}$ by complex scalars, which reduces (after imposing all the properties that must be satisfied) to define 'multiplication' by i. That is, we need to choose an $\mathbb{R}$-linear map J, and define $c * z = \operatorname{Re} c\ z + \operatorname{Im} c\ J(z)$ for any $c, z \in \mathbb{C}$. It follows from imposing $\mathrm{i}^2 * z = \mathrm{i} * \mathrm{i} * z$, that $J(J(z)) = -z$. The structure induced by L is defined by choosing $J = L^{-1} \circ I \circ L$, where $I(z) = \mathrm{i}\, z$ in the standard way.

We will end this linear discussion by considering how Beltrami coefficients and dilatations change under inversion and composition of linear maps. We start with inversion. Given a map L as above, it is easy to check that

$$L^{-1}(w) = \frac{1}{|a|^2 - |b|^2}(\overline{a}w - b\overline{w}).$$

It follows that

$$\mu(L^{-1}) = -\mu(L)e^{\mathrm{i}\,(2\arg a)}, \tag{1.1}$$

and hence $|\mu(L^{-1})| = |\mu(L)|$, which implies that

$$K(L^{-1}) = K(L). \tag{1.2}$$

Now suppose $j \in \{1, 2\}$ and we have two $\mathbb{R}$-linear maps $L_j(z) = a_j z + b_j \overline{z}$ with dilatation K_j and Beltrami coefficient μ_j. The ellipse defined by the composition $L_1 \circ L_2$ is the preimage under L_2 of the ellipse defined by the map L_1. Observe that from linear algebra, we can assure that

$$K(L_1 \circ L_2) \leqslant K(L_1)K(L_2),$$

since the maximal possible stretch is the product of the two maximal stretches of each of the maps, while the corresponding holds for the minimal stretches.

If we want to know the Beltrami coefficient for this new ellipse we compute the composition

$$(L_1 \circ L_2)(z) = (a_1 a_2 + b_1 \overline{b_2})z + (a_1 b_2 + b_1 \overline{a_2})\overline{z},$$