DISCRETE SYSTEMS AND INTEGRABILITY

This first introductory text to discrete integrable systems introduces key notions of integrability from the vantage point of discrete systems, also making connections with the continuous theory where relevant. While treating the material at an elementary level, the book also highlights many recent developments. Topics include: Darboux and Bäcklund transformations, difference equations and special functions, multidimensional consistency of integrable lattice equations, associated linear problems (Lax pairs), connections with Padé approximants and convergence algorithms, singularities and geometry, Hirota's bilinear formalism for lattices, intriguing properties of discrete Painlevé equations and the novel theory of Lagrangian multiforms. The book builds the material in an organic way, emphasizing interconnections between the various approaches, while the exposition is mostly done through explicit computations on key examples. Written by respected experts in the field, the book's numerous exercises and thorough list of references will benefit both upper-level undergraduate and beginning graduate students as well as researchers from other disciplines.

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To Marja, Robert and Alain

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Preface

There has been a surge of interest in discrete integrable systems in the last two decades. The term "discrete integrable systems" (DIS) combines two aspects: discreteness and integrability. The subtle concept of integrability touches on global existence and regularity of solutions, explicit solvability, as well as compatibility and consistency – fundamental features, which form the recurrent themes of this book. On the other hand, by discrete systems we mean mathematical models that involve *finite* (as opposed to *infinitesimal*) operations. In a sense, discrete systems are essential to an understanding of integrability, and this book serves to provide an introduction to integrability from the perspective of discrete systems.

Discrete integrable systems include many types of equations, such as recurrence relations, difference equations and dynamical mappings as well as equations that contain a mixture of derivative and difference operators. Integrable systems have appeared throughout the history of mathematics without being recognized as integrable. An example is the equation that arises from the geometric collinearity theorem of Menelaus of Alexandria in the second century. Other examples are found in the defining equations of classical and nonclassical special functions. Important physical models, such as the equations of motion of the Euler top, are also integrable. The elliptic billiard is a classic example of a DIS, and the corresponding geometric result is the *Poncelet's porism*. It is, however, only in the second half of the twentieth century that the subject of integrable systems has grown as a discipline in its own right, and only in the last roughly two decades that the field of DIS has come to prominence as an area in which numerous breakthroughs have taken place, inspiring new developments in other areas of mathematics.

The number of integrable systems is large and growing, and the list of them includes a large number of examples that have application to physics and other scientific fields. In many cases, it turns out that one and the same discrete equation may be interpreted in different ways: as a dynamical map, as a difference equation and as an addition formula. This means that in the study of a given DIS a large number of branches of mathematics come together. Moreover, integrable equations are not necessarily isolated objects, but have many mathematical interconnections between each other. Highlighting these interconnections is another aim of this book.

Integrable systems are both rare and universal. Like prime numbers, their rarity is due to the intricate mathematical structures underlying them, while at the same time these structures explain their universality. This book aims to describe their underlying structures, but there are still many that remain to be discovered.

This book grew from lectures given by the authors in courses on DIS for students at the advanced undergraduate or beginning graduate level, at the University of Leeds, the University of Turku and the University of Sydney. The presentation of the book is pitched at a level suitable for students as well as researchers who are seeking an introduction to the subject.

The exposition of the book will be led by exploring key examples. We will show how the major features of integrability arise in an organic way by revealing them through explicit calculations. Each chapter contains several exercises, which will further illustrate the methods and expand on the main text. The initial chapters contain the basic material. Additional specialized or advanced topics are provided in the later chapters. Some important background material on difference calculus, elliptic functions and determinantal identities are provided in the appendices.

We start in Chapter 1 with an overview of the types of difference equations we will encounter, highlighting integrable systems as examples. In Chapter 2 we show how discrete systems, in the form of difference equations, arise naturally from known continuous equations in two different settings: the linear case (with difference equations arising from the recurrence structure of families of special functions) and the nonlinear case (addition rules for elliptic functions and Bäcklund–Schlesinger transformations for Painlevé transcendents). In the multivariable case we show how integrable partial difference equations arise as permutability conditions from Bäcklund transforms of soliton equations. This emergence of fully discrete equations (also called *lattice equations*) leads naturally to a characterization of integrability by *multidimensional consistency*. This property is explored in depth in Chapter 3, where we study its consequences and also discuss additional sources of DIS. Chapter 4 provides an interlude from the main theory by illustrating the links between integrable systems and applications in the theory of numerical algorithms. The circle is closed in Chapter 5 by providing continuum limits that lead back to continuous systems from the discrete ones. In the remaining chapters we will delve deeper into the theory and explore reductions and solutions, as well as other characteristics of integrability. In Chapter 6, we shift the focus to ordinary difference equations and dynamical mappings and provide various points of view to study them. The problem of detecting and identifying integrable difference equations by methods based on their singularity structure and growth is discussed in Chapter 7. Chapters 8 and 9 discuss two different approaches to the construction of special solutions of the partial difference equations encountered in the earlier chapters. First, in Chapter 8, we present and explain Hirota's bilinear method: in particular, how it can be used to obtain soliton solutions. In Chapter 9, another approach to soliton solutions is provided by the Cauchy matrix approach, which allows us to see that many seemingly different systems are closely interrelated. Other types of solutions are obtained in Chapter 10 through similarity reductions of partial difference equations. These are transcendental solutions obtained as solutions of a class of nonautonomous nonlinear ordinary difference equations, called the discrete Painlevé equations. The discrete Painlevé equations have very deep mathematical properties, which are described in Chapter 11. The final chapter, Chapter 12, provides insight into the geometrical aspects of integrable systems through their Lagrangian structure and the corresponding new variational approach suitable for integrable systems.

The book is by no means intended to be an exhaustive treatment of all topics related to discrete integrability, but rather reflects the authors' interests and experience. For example, we do not give a comprehensive treatment of Lie symmetries and conservation laws of discrete systems; for that, we refer to Hydon (2014). Other topics we omit are ultra-discrete equations and tropical geometry, as well as discrete quantum integrable systems. Currently, there exist only a few monographs on DIS: Suris (2003) focuses primarily on integrable discretizations from a Hamiltonian perspective, while Bobenko and Suris (2008) consider differential geometric aspects; Duistermaat (2010) deals mainly with the QRT (Quispel–Roberts–Thompson) dynamical mapping and the algebraic geometry of the associated elliptic surfaces.

We would like to draw the reader's attention also to the biennial conference series SIDE, devoted to *Symmetries and Integrability of Difference Equations*, the proceedings of which provide a good account of the state of the art of the subject. For further details, see `http://side-conferences.net`. We would also like to mention some collections of lectures on DIS; for example, Grammaticos et al. (2004) and Levi et al. (2011).

Many colleagues and students have contributed to the evolution of this book in various ways. We would in particular like to thank (in alphabetical order) James Atkinson, Sam Butler, Neslihan Delice, Chris Field, Wei Fu, Basil Grammaticos, Mike Hay, Anthony Henderson, Phil Howes, Paul Jennings, Pavlos Kassotakis, Sotiris Konstantinou-Rizos, Sarah Lobb, Nobutaka Nakazono, Maciej Nieszporski, Chris Ormerod, Vassilis Papageorgiou, Reinout Quispel, Alfred Ramani, John Roberts, Yang Shi, Ying Shi, Paul Spicer, Tomoyuki Takenawa, Tasos Tongas, Dinh Tran, Peter van der Kamp, Claude Viallet, Pavlos Xenitidis, Sikarin Yoo-Kong, Da-jun Zhang and Songlin Zhao. Special thanks go to Da-jun Zhang for his careful and detailed reading of the galley proofs.