Cambridge University Press
978-1-107-04259-9 - The Banach–Tarski Paradox: Second Edition
Grzegorz Tomkowicz and Stan Wagon
Frontmatter
More information

THE BANACH–TARSKI PARADOX

Second Edition

The Banach–Tarski Paradox is a most striking mathematical construction: it asserts that a solid ball can be taken apart into finitely many pieces that can be rearranged using rigid motions to form a ball twice as large. This volume explores the consequences of the paradox for measure theory and its connections with group theory, geometry, set theory, and logic.

This new edition of a classic book unifies contemporary research on the paradox. It has been updated with many new proofs and results and discussions of the many problems that remain unsolved. Among the new results presented are several unusual paradoxes in the hyperbolic plane, one of which involves the shapes of Escher's famous "Angel and Devils" woodcut. A new chapter is devoted to a complete proof of the remarkable result that the circle can be squared using set theory, a problem that had been open for over sixty years.

Grzegorz Tomkowicz is a self-educated Polish mathematician who has made several important contributions to the theory of paradoxical decompositions and invariant measures.

Stan Wagon is a Professor of Mathematics at Macalester College. He is a winner of the Wolfram Research Innovator Award, as well as numerous writing awards including the Ford, Evans, and Allendoerfer Awards. His previous work includes *A Course in Computational Number Theory* (2000), *The SIAM 100-Digit Challenge* (2004) and *Mathematica® in Action* (3rd Ed. 2010).

Encyclopedia of Mathematics and its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

Encyclopedia of Mathematics and its Applications

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

www.cambridge.org/mathematics.

114 J. Beck *Combinatorial Games*
115 L. Barreira and Y. Pesin *Nonuniform Hyperbolicity*
116 D. Z. Arov and H. Dym *J-Contractive Matrix Valued Functions and Related Topics*
117 R. Glowinski, J.-L. Lions, and J. He *Exact and Approximate Controllability for Distributed Parameter Systems*
118 A. A. Borovkov and K. A. Borovkov *Asymptotic Analysis of Random Walks*
119 M. Deza and M. Dutour Sikirić *Geometry of Chemical Graphs*
120 T. Nishiura *Absolute Measurable Spaces*
121 M. Prest *Purity, Spectra and Localisation*
122 S. Khrushchev *Orthogonal Polynomials and Continued Fractions*
123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivity*
124 F. W. King *Hilbert Transforms I*
125 F. W. King *Hilbert Transforms II*
126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
127 M. Grabisch *et al. Aggregation Functions*
128 L. W. Beineke and R. J. Wilson (eds.) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
129 J. Berstel, D. Perrin, and C. Reutenauer *Codes and Automata*
130 T. G. Faticoni *Modules over Endomorphism Rings*
131 H. Morimoto *Stochastic Control and Mathematical Modeling*
132 G. Schmidt *Relational Mathematics*
133 P. Kornerup and D. W. Matula *Finite Precision Number Systems and Arithmetic*
134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
136 A. Kristály, V. D. Rădulescu, and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar, and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc, and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov, and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein *et al. Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn–Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal, and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. M. Bisci, V. Radulescu, and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 G. Tomkowicz and S. Wagon *The Banach–Tarski Paradox (Second Edition)*

Encyclopedia of Mathematics and its Applications

The Banach–Tarski Paradox

Second Edition

GRZEGORZ TOMKOWICZ
Centrum Edukacji G2, Bytom, Poland

STAN WAGON
Macalester College, St. Paul, Minnesota

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

One Liberty Plaza, New York, NY 10006

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107042599

First published 2016

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data
Names: Tomkowicz, Grzegorz. | Wagon, S.
Title: The Banach-Tarski paradox.
Description: Second edition / Grzegorz Tomkowicz, Centrum Edukacji G2, Bytom, Poland, Stan Wagon, Macalester College, St. Paul, Minnesota. | New York, NY : Cambridge University Press, [2016] | Series: Encyclopedia of mathematics and its applications ; 163 | Previous edition: The Banach-Tarski paradox / Stan Wagon (Cambridge : Cambridge University Press, 1985). | Includes bibliographical references and index.
Identifiers: LCCN 2015046410 | ISBN 9781107042599 (hardback : alk. paper)
Subjects: LCSH: Banach-Tarski paradox. | Measure theory. | Decomposition (Mathematics)
Classification: LCC QA248 .W22 2016 | DDC 511.3/22–dc23
LC record available at http://lccn.loc.gov/2015046410

ISBN 978-1-107-04259-9 Hardback

To Jan Mycielski, whose enthusiasm and knowledge strongly influenced both of us

Cambridge University Press
978-1-107-04259-9 - The Banach–Tarski Paradox: Second Edition
Grzegorz Tomkowicz and Stan Wagon
Frontmatter
More information

DELIANS: *How can we be rid of the plague?*
DELPHIC ORACLE: *Construct a cubic altar having double the size of the existing one.*
BANACH AND TARSKI: *Can we use the Axiom of Choice?*

Contents

Cambridge University Press
978-1-107-04259-9 - The Banach–Tarski Paradox: Second Edition
Grzegorz Tomkowicz and Stan Wagon
Frontmatter
More information

Foreword

This book is motivated by the following theorem of Hausdorff, Banach, and Tarski: Given any two bounded sets A and B in three-dimensional space $\mathbb{R}^3$, each having nonempty interior, one can partition A into finitely many disjoint parts and rearrange them by rigid motions to form B. This, I believe, is the most surprising result of theoretical mathematics. It shows the imaginary character of the unrestricted idea of a set in $\mathbb{R}^3$. It precludes the existence of finitely additive, congruence-invariant measures over all bounded subsets of $\mathbb{R}^3$, and it shows the necessity of more restricted constructions, such as Lebesgue measure.

In the 1950s, the years of my mathematical education in Poland, this result was often discussed. J. F. Adams, T. J. Dekker, J. von Neumann, R. M. Robinson, and W. Sierpiński wrote about it; my PhD thesis was motivated by it. (All this is referenced in this book.) Thus it is a great pleasure to introduce you to this book, where this striking theorem and many related results in geometry and measure theory, and the underlying tools of group theory, are presented with care and enthusiasm. The reader will also find some applications of the most recent advances of group theory to measure theory: the work of Gromov, Margulis, Rosenblatt, Sullivan, Tits, and others.

But to me the interest of mathematics lies no more in its theorems and theories than in the challenge of its surprising problems. And, on the pages of this book, you will find many old and new open problems. So let me conclude this foreword by turning your attention to one of them, from my teacher E. Marczewski (before 1939, he published under the name Szpilrajn): Does there exist a finite sequence $A_1, \ldots, A_n$ of pairwise disjoint open subsets of the unit cube and isometries $\sigma_1, \ldots, \sigma_n$ of $\mathbb{R}^3$ such that the unit cube is a proper subset of the topological closure of the union $\sigma_1(A_1) \cup \ldots \cup \sigma_n(A_n)$? This remarkable problem is discussed in Chapters 3, 9, 11, and 13 of this book.

I wish you the most pleasant reading and many fruitful thoughts.

Addendum to the Foreword

Several spectacular results have been proved since the 1985 first edition of this book. Two of them are particularly striking.

A. The answer to Marczewski's problem mentioned earlier is yes. R. Dougherty and M. Foreman have shown (Thm. 11.16) that any two bounded nonempty open sets A and B in the Euclidean space $\mathbb{R}^n$ ($n \geq 3$) are equivalent in the following sense: A has finitely many disjoint regular-open subsets whose union is everywhere dense in A and which can be moved by isometries into disjoint subsets of B whose union is everywhere dense in B. Similar results hold for spheres $\mathbb{S}^n$ and the hyperbolic spaces $\mathbb{H}^n$ ($n \geq 2$).

B. The answer to Tarski's "squaring the circle" problem is also yes. M. Laczkovich proved that if a circle and a square in $\mathbb{R}^2$ have the same area, then they are equivalent by finite decomposition, and the isometries of the corresponding pieces are translations (i.e., simple vector addition). And the same is true for many other pairs of sets in $\mathbb{R}^n$. A proof of this is in Chapter 9.

Some outstanding problems are still open:

1. **The Banach–Ulam problem 2 from the Scottish Book.** Does every compact metric space admit a finitely additive, congruence-invariant probability measure on its Borel sets? (Question 3.13)
2. **Exotic Borel measures.** Is Lebesgue measure the only finitely additive, isometry-invariant measure on the Borel sets of $\mathbb{R}^n$ that normalizes the unit cube? (It is not the only translation-invariant one. In the minimal model for set theory that contains all real and all ordinal numbers (usually called $L(\mathbb{R})$), and assuming a certain large cardinal exists, all sets are Lebesgue measurable and have the Property of Baire, and in this model the answer is yes.) (Question 13.13)

3. **Borel circle-squaring.** Can the pieces of the circle-squaring decomposition of Laczkovich be taken to be Borel sets? (§9.3)

All these results and problems are presented in a penetrating and lucid way in this new edition.

Jan Mycielski
Boulder, Colorado
August 2015

Preface

Although many properties of infinite sets and their subsets were considered to be paradoxical when they were discovered, the development of paradoxical decompositions really began with the formalization of measure theory at the beginning of the twentieth century. The classic example (Vitali, 1905) of a non–Lebesgue measurable set was the first instance of the use of a paradoxical decomposition to show the nonexistence of a certain type of measure. Ten years later, Hausdorff constructed a much more surprising paradox on the surface of the sphere (again, to show the nonexistence of a measure), and this inspired some important work in the 1920s. Namely, there was Banach's construction of invariant measures on the line and in the plane (which required the discovery of the main ideas of the Hahn–Banach Theorem) and the famous Banach–Tarski Paradox on duplicating, or enlarging, spheres and balls. This latter result, which at first seems patently impossible, is often stated as follows: It is possible to cut up a pea into finitely many pieces that can be rearranged to form a ball the size of the sun!

Their construction has turned out to be much more than a curiosity. Ideas arising from the Banach–Tarski Paradox have become the foundation of a theory of finitely additive measures, a theory that involves much interplay between analysis (measure theory and linear functionals), algebra (combinatorial group theory), geometry (isometry groups), and topology (locally compact topological groups). Moreover, the Banach–Tarski Paradox itself has been useful in important work on the uniqueness of Lebesgue measure: It shows that certain measures necessarily vanish on the sets of Lebesgue measure zero.

The purpose of this volume is twofold. The first aim is to present proofs that are as simple as possible of the two main classical results—the Banach–Tarski Paradox in $\mathbb{R}^3$ (and $\mathbb{R}^n$, $n > 3$) and Banach's theorem that no such paradox exists in $\mathbb{R}^1$ or $\mathbb{R}^2$. The first three chapters are devoted to the paradox and are accessible to anyone familiar with the rudiments of linear algebra, group theory, and countable sets. (Background related to the Euclidean isometry groups is included in Appendix A.) Chapter 12, which contains Banach's theorem in $\mathbb{R}^1$ and $\mathbb{R}^2$, can be read independently of Chapters 4–11 but requires a little more background in measure theory (Lebesgue measure) and general topology (Tychonoff Compactness Theorem). Although isolated proofs use some special techniques, such as transfinite induction or analytic functions, most of the material is accessible to a first- or second-year graduate student.

The book's other purpose is to serve as a background source for those interested in current research that has a connection to paradoxical decompositions. The period since 1980 has been especially active, and several classic problems in this area have been solved, some by using the deepest techniques of modern mathematics. This volume contains a unified and modern treatment of the fundamental results about amenable groups, finitely additive measures, and free groups of isometries and so should prove useful to someone in any field who is interested in these modern results and their historical context.

The group theory connections arise from the difference between the isometry groups of $\mathbb{R}^2$ and $\mathbb{R}^3$, a difference that explains the presence of the Banach–Tarski Paradox in $\mathbb{R}^3$ and its absence in the plane. This distinction led to the study of the class of groups that are not paradoxical, that is, groups that cannot be duplicated by left translation of finitely many pairwise disjoint subsets. This class, denoted by *AG* for amenable groups, contains all solvable and finite groups but excludes free non-Abelian groups. A famous problem is whether *AG* equals *NF*, the class of groups not having a free non-Abelian subgroup. This was solved in 1980 (Ol'shanskii [Ols80]), using ideas connected with growth conditions in groups (Cohen [Coh82]) and the solution of Burnside's Problem (Adian [Adi79]). However, the classes *AG* and *NF* do coincide when restricted to linear groups (a deep result of Tits [Tit72]) or to connected, locally compact topological groups (Balcerzyk and Mycielski [BM57]). Growth conditions in groups, first studied in depth by Milnor and Wolf [Mil68b, Mil68c, Wol68], also elucidate a weaker sort of paradox, the Sierpiński–Mazurkiewicz Paradox, which exists in $\mathbb{R}^2$ but not in $\mathbb{R}^1$. The class *AG* has also led to the study of (topological) amenability in topological groups (where only Borel sets are considered). Amenability and the related notion of an invariant mean have proven to be useful tools in the study of topological groups (Greenleaf [Gre69]). Chapter 12 contains an introduction to the theory of amenable groups, and Chapter 14 discusses the relevance of growth conditions for the theory of amenability and paradoxical decompositions.

In analysis, important work solving the Ruziewicz Problem has its roots in Banach's results about $\mathbb{R}^1$ and $\mathbb{R}^2$. Banach showed that Lebesgue measure is not the only finitely additive, isometry-invariant measure on the bounded, measurable subsets of the plane (or line) that normalizes the unit square (or interval). The analogous problem for $\mathbb{R}^3$ and beyond was unsolved for over fifty years. But using Kazhdan's Property T and techniques of functional analysis, Margulis [Mar80, Mar82], Sullivan [Sul81], Rosenblatt [Ros81], and Drinfeld [Dri85] settled this question in the expected way: No "exotic" measures exist, except in the cases Banach considered. The construction of exotic measures in $\mathbb{R}^1$ and $\mathbb{R}^2$ and a discussion of the higher-dimensional situation are presented in Chapter 13.

Actions of free groups are central to the whole theory, and a general problem is to determine which naturally occurring groups have certain sorts of free subgroups. Chapters 4, 6, 7, and 8 present the classical results on the isometry groups of spheres, Euclidean spaces. Some problems were solved relatively recently. Until the work of Deligne and Sullivan [DS83], it was not known that $SO_6(\mathbb{R})$ (or $SO_{4n+2}(\mathbb{R})$) contained a free non-Abelian subgroup, no element of which (except the identity) has $+1$ as an eigenvalue. And a similar problem about locally commutative free subgroups was solved in all $SO_n(\mathbb{R})$, except $SO_5(\mathbb{R})$, in 1956 (Dekker [Dek56b]), with the remaining case solved by A. Borel [Bor83] in a paper generalizing the work of Deligne and Sullivan. Chapter 7 presents a technique for improving this type of result to get uncountable free subgroups and discusses the geometrical consequences of these larger free groups of isometries.

There has been a remarkable amount of progress on famous problems since the first edition of this book appeared in 1985. The most striking discoveries are the solutions to two very famous open questions. Laczkovich solved the famous Tarski circle-squaring problem by showing that a disk and square in the plane having the same area are equidecomposable; and Dougherty and Foreman proved that a Banach–Tarski-type paradox exists with pieces having the Property of Baire. Other noteworthy results are T. Wilson's solution to the de Groot problem—he showed that the pieces in the classic paradox could be chosen so that the moves to the new positions preserve disjointness at every instant—and the work of Sherman and Just on bounded paradoxical sets in the plane. This new edition contains all the essential details of the work of Laczkovich, Wilson, and Sherman. Another addition is a presentation of Følner's Condition and amenability through the use of pseudogroups in Chapter 12.

Also new are Chapter 4, which collects diverse results about the hyperbolic plane, and Chapter 8, which focuses on the Euclidean plane and the group of area-preserving linear transformations. A main theme in many of these is that counterintuitive paradoxes can be constructed without requiring the Axiom of Choice.

The book is divided into two parts. The first deals with the construction of paradoxical decompositions (which imply that certain sorts of finitely additive measures do not exist), and the second deals with the construction of measures (which show why certain paradoxical decompositions do not exist). Chapter 11 ties the two parts together by presenting a theorem of Tarski that asserts that the existence of a paradoxical decomposition is equivalent to the nonexistence of an invariant, finitely additive measure. The final chapter, Chapter 15, discusses some technical and philosophical points relevant to the foundational discussion engendered by the use of the Axiom of Choice in the Banach–Tarski Paradox and related results.

Here is an observation of Bertrand Russell from 1918 [Rus10]: "The point of philosophy is to start with something so simple as not to seem worth stating, and to end with something so paradoxical that no one will believe it." Although paradoxes are by no means the point of the study of mathematics, there is no question that counterintuitive results can clarify our understanding and provide motivation for more detailed study. The Banach–Tarski Paradox certainly plays such a role, and we both have found its study to be intensely rewarding. We hope you enjoy learning about it.

The authors would like to express their gratitude to Jan Mycielski and Joseph Rosenblatt for their support and advice during the preparation of this volume. Moreover, a book such as this, touching on many mathematical disciplines, would not have been possible without the willingness of many people to share their expertise. The help of the following mathematicians is gratefully acknowledged: Bill Barker, Curtis Bennett, Armand Borel, Rotislav Grigorchuk, Branko Grünbaum, William Hanf, Joan Hutchinson, Victor Klee, Miklos Laczkovich, Rich Laver, Robert Macrae, Dave Morris, Arlan Ramsay, Robert Riley, Robert Solovay, Dennis Sullivan, Alan Taylor, and Trevor Wilson.

Grzegorz Tomkowicz	Stan Wagon
Centrum Edukacji G2	Macalester College
Moniuszki 9	1600 Grand Avenue
41-902 Bytom, Poland	St. Paul, Minnesota, USA
gtomko@vp.pl	wagon@macalester.edu