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Mark S. Joshi and Jane M. Paterson  
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## Introduction to Mathematical Portfolio Theory

In this concise yet comprehensive guide to the mathematics of modern portfolio theory, the authors discuss mean–variance analysis, factor models, utility theory, stochastic dominance, very long term investing, the capital asset pricing model, risk measures including VAR, coherence, market efficiency, rationality and the modelling of actuarial liabilities. Each topic is clearly explained with assumptions, mathematics, limitations, problems and solutions presented in turn.

Joshi's trademark style of clarity and practicality is here brought to classical financial mathematics. The book is suitable for mathematically trained students in actuarial studies, business and economics as well as mathematics and finance, and it can be used both for self-study and as a course text. The authors' experience as both academics and practitioners brings clarity and relevance to the book, whilst ensuring that the limitations of models are highlighted.

MARK S. JOSHI is a researcher and consultant in mathematical finance, and a Professor at the University of Melbourne. His research focuses on derivatives pricing and interest rate derivatives in particular. He is the author of numerous research articles on quantitative finance and four books.

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CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town,  
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9781107042315](http://www.cambridge.org/9781107042315)

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First published 2013

Printed and bound in the United Kingdom by Bell and Bain Ltd

*A catalogue record for this publication is available from the British Library*

ISBN 978-1-107-04231-5 Hardback

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978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

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Frontmatter

[More information](#)

## Contents

	<i>Preface</i>	<i>page xi</i>
<b>1</b>	<b>Definitions of risk and return</b>	<b>1</b>
	1.1 Introduction	1
	1.2 Measuring return	2
	1.3 Portfolio constraints	3
	1.4 Defining risk with variance	5
	1.5 Other risk measures	9
	1.6 Review	9
	1.7 Problems	10
<b>2</b>	<b>Efficient portfolios: the two-asset case</b>	<b>12</b>
	2.1 Defining efficiency	12
	2.2 Two-asset portfolios	13
	2.2.1 The effect of correlation	15
	2.2.2 Classifying the curves	20
	2.3 Review	21
	2.4 Problems	21
<b>3</b>	<b>Portfolios with a risk-free asset</b>	<b>24</b>
	3.1 The risk-free asset	24
	3.2 Efficiency with a risk-free asset	26
	3.3 Tangent portfolios	29
	3.4 Examples	30
	3.5 Borrowing restrictions	34
	3.6 Review	36
	3.7 Problems	36
<b>4</b>	<b>Finding the efficient frontier – the multi-asset case</b>	<b>39</b>
	4.1 Finding the tangent portfolio	39
	4.2 Geometry of the frontier	40

Cambridge University Press

978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

Mark S. Joshi and Jane M. Paterson

Frontmatter

[More information](#)

vi

*Contents*

4.3	The minimal variance portfolio	42
4.4	Illustrating the method	43
4.5	The derivation of the algorithm	44
4.6	Solution via Lagrange multipliers	52
4.7	Review	53
4.8	Problems	54
<b>5</b>	<b>Single-factor models</b>	<b>57</b>
5.1	Introduction	57
5.2	Mathematical formulation of the single-factor model	58
5.3	Data requirements for the single-factor model	59
5.4	Understanding beta	60
5.5	Techniques for parameter estimation	62
5.6	Assessing estimates	64
5.7	Portfolio betas	67
5.8	Blume's technique	67
5.9	Fundamental analysis	70
5.10	Review	71
5.11	Problems	72
<b>6</b>	<b>Multi-factor models</b>	<b>75</b>
6.1	Mathematical formulation	75
6.2	Types of multi-factor models	78
6.3	Orthogonalisation for multi-factor models	79
6.4	Review	84
6.5	Problems	84
<b>7</b>	<b>Introducing utility</b>	<b>88</b>
7.1	Limitations of mean–variance analysis	88
7.2	Defining utility	90
7.3	Properties of utility functions	91
7.4	Quadratic utility and portfolio theory	93
7.5	Indifference curves	94
7.6	Approximating with quadratic utility	95
7.7	Indifference pricing	96
7.8	Review	98
7.9	Problems	98
<b>8</b>	<b>Utility and risk aversion</b>	<b>102</b>
8.1	Risk aversion and curvature	102
8.2	Absolute risk aversion	103
8.3	Relative risk aversion	105
8.4	Varying the utility function	107

Cambridge University Press

978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

Mark S. Joshi and Jane M. Paterson

Frontmatter

[More information](#)

<i>Contents</i>		vii
8.5	St Petersburg revisited	109
8.6	Review	110
8.7	Problems	110
<b>9</b>	<b>Foundations of utility theory</b>	<b>113</b>
9.1	Analysing utility theory through experimental economics	113
9.2	The rational investor	115
9.3	The rational expectations theorem	117
9.4	Review	121
9.5	Problems	121
<b>10</b>	<b>Maximising long-term growth</b>	<b>122</b>
10.1	Geometric means	122
10.2	Kelly's theorem	125
10.3	Review	130
10.4	Problems	130
<b>11</b>	<b>Stochastic dominance</b>	<b>133</b>
11.1	Introduction	133
11.2	Dominance	133
11.3	First-order stochastic dominance	134
11.4	Second-order stochastic dominance	138
11.5	Review	145
11.6	Problems	145
<b>12</b>	<b>Risk measures</b>	<b>148</b>
12.1	Introduction	148
12.2	Value-at-Risk	149
12.3	Computing VAR	152
12.4	VAR estimates and excesses	154
12.5	Evaluating risk measures	154
12.6	Other risk measures and the axioms	158
12.7	Conditional expected shortfall	160
12.8	CES and the coherence axioms	162
12.9	Risk measures and utility	165
12.10	Economic capital modelling	165
12.11	Review	166
12.12	Problems	167
12.13	Additional problems	168
<b>13</b>	<b>The Capital Asset Pricing Model</b>	<b>169</b>
13.1	Introduction	169
13.2	From tangent to market	169

Cambridge University Press

978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

Mark S. Joshi and Jane M. Paterson

Frontmatter

[More information](#)

viii

*Contents*

13.3	Assessing the CAPM assumptions	173
13.4	Using CAPM	173
13.5	Implementing CAPM	173
13.6	Eliminating the risk-free asset	174
13.7	Testing CAPM	176
13.8	Roll's objection	178
13.9	Review	179
13.10	Problems	180
<b>14</b>	<b>The arbitrage pricing model</b>	<b>182</b>
14.1	Introduction	182
14.2	Defining arbitrage	182
14.3	The one-step binomial tree	183
14.4	The principle of no arbitrage	184
14.5	Using replication to price a call option	184
14.6	Risk-neutrality	185
14.7	Interest rates and discounting	186
14.8	The trinomial tree and limitations of no arbitrage	188
14.9	Arbitrage and randomness	189
14.10	Arbitrage Pricing Theory	190
14.11	Computations	192
14.12	An alternative approach to computation	196
14.13	Introducing realism	197
14.14	APT versus CAPM	197
14.15	APT in practice	198
14.16	Applications of APT	199
14.17	Criticising APT	199
14.18	Review	200
14.19	Problems	200
<b>15</b>	<b>Market efficiency and rationality</b>	<b>203</b>
15.1	Introduction	203
15.2	Defining efficiency	203
15.3	Testing efficiency	206
15.4	Anomalies	207
15.5	Conclusions on efficiency	209
15.6	Rationality	210
15.7	Famous bubbles	211
15.8	Justifying high stock prices	213
15.9	Further reading	213
15.10	Review	213



Cambridge University Press

978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

Mark S. Joshi and Jane M. Paterson

Frontmatter

[More information](#)*Contents*

ix

15.11	Questions	214
<b>16</b>	<b>Brownian motion and stock price models across time</b>	<b>215</b>
16.1	Introduction	215
16.2	Brownian motion	215
16.3	Differentiability properties of Brownian motion	216
16.4	Computing with Brownian motion	219
16.5	More properties	220
16.6	Arithmetic and geometric Brownian motions	222
16.7	Log-normal models for stock prices	224
16.8	Auto-regressive processes	226
16.9	The Wilkie model	227
16.10	Using the Wilkie model	230
16.11	Review	231
16.12	Questions	232
<i>Appendix A</i>	<b>Matrix algebra</b>	234
<i>Appendix B</i>	<b>Solutions</b>	238
	<i>References</i>	309
	<i>Index</i>	311

Cambridge University Press

978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

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Frontmatter

[More information](#)

---

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978-1-107-04231-5 - Introduction to Mathematical Portfolio Theory

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Frontmatter

[More information](#)

## Preface

This book grew out of a lecture course taught at the University of Melbourne over a series of years. The audience was third-year actuarial students who partially gained an exemption from the Faculty and Institute of Actuaries CT8 module if they did well. The nature of the audience and the exemption placed certain constraints on the syllabus and delivery that made it hard to find a suitable textbook. The graduate level texts simply being too hard, whilst the undergraduate and MBA books did not cover the mathematics in sufficient depth. In particular, the students were fairly mathematical but more oriented towards computations than proof. In addition, the choice of topics had to be tuned to the actuarial syllabus and that is reflected in this book.

In terms of mathematical level, we strive to achieve a mid-level where mathematics is not shied away from nor hived off to appendices, but also not so hard as to deter the undergraduate reader. Also, this being a book on mathematical portfolio theory, the mathematics takes centre stage for most of the book: our objective is to study the mathematics of portfolio theory without losing sight of the finance. As both authors have been both practitioners and academics, a theme throughout is that a model is a model and not reality, and we aim to highlight our assumptions and their consequences. We provide a lot of problems with solutions in the belief that this is ultimately how the material is best learnt, and as a consequence of the fact that students always want more problems and more solutions.

We first look at the definitions of risk and return. We then explore Markowitz's portfolio theory. We start with the two-asset case, then add a riskless asset, and finally treat the general case. We derive a couple of different ways to find efficient portfolios in that case. We then move on to seeing how simplifying correlation structures can help to reduce the amount of data needed to estimate the model parameters.

We then make a long excursion into utility theory, looking at both its pros

and cons. We also look at the offshoots of stochastic dominance and geometric mean maximisation. An important issue in any insurance company or bank is risk control, and we therefore look at risk measures including VAR and conditional expected shortfall. We also examine the coherence axioms.

We then move onto a critical look at the capital asset pricing model and the arbitrage pricing theory. We also discuss market efficiency and rationality. Here we adopt a more discursive viewpoint. We finish by looking at long-term models of stock prices using Brownian motion and the Wilkie model.

Much of this book has been shaped by interactions with our students and from their explicit feedback, and we thank them all for their input. We particularly thank Timothy Hillman for his detailed comments on the manuscript.

For the reader whose appetite has been whetted and wishes to study the material in greater depth, we mention a few books which we have found helpful.

- Elton *et al.*, [6], is very discursive and contains very detailed references but is much less mathematical than this book.
- Pennachi, [15], is a nice text aimed at PhD students.
- Cochrane, [3], has both good discussion and good mathematics and will reward the reader who perseveres.
- Markowitz's original book, *Portfolio Selection*, [11], is still a good read.

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Melbourne 2012