MANIFOLDS, TENSORS, AND FORMS

Providing a succinct yet comprehensive treatment of the essentials of modern differential geometry and topology, this book's clear prose and informal style make it accessible to advanced undergraduate and graduate students in mathematics and the physical sciences.

The text covers the basics of multilinear algebra, differentiation and integration on manifolds, Lie groups and Lie algebras, homotopy and de Rham cohomology, homology, vector bundles, Riemannian and pseudo-Riemannian geometry, and degree theory. It also features over 250 detailed exercises, and a variety of applications revealing fundamental connections to classical mechanics, electromagnetism (including circuit theory), general relativity, and gauge theory. Solutions to the problems are available for instructors at www.cambridge.org/9781107042193.

Paul Renteln is Professor of Physics in the Department of Physics, California State University San Bernardino, where he has taught a wide range of courses in physics. He is also Visiting Associate in Mathematics at the California Institute of Technology, where he conducts research into combinatorics.

MANIFOLDS, TENSORS, AND FORMS

An Introduction for Mathematicians and Physicists

PAUL RENTELN

California State University San Bernardino
and
California Institute of Technology

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Published in the United States of America by Cambridge University Press, New York

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107042193

First published 2014

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Renteln, Paul, 1959– author.
Manifolds, tensors, and forms : an introduction for mathematicians and physicists / Paul Renteln.
pages cm
Includes bibliographical references and index.
ISBN 978-1-107-04219-3 (alk. paper)
1. Geometry, Differential – Textbooks. 2. Manifolds (Mathematics) – Textbooks.
3. Calculus of tensors – Textbooks. 4. Forms (Mathematics) – Textbooks. I. Title.
QA641.R46 2013
516.3′6–dc23
2013036056

ISBN 978-1-107-04219-3 Hardback

Additional resources for this publication at www.cambridge.org/9781107042193

Cambridge University Press
978-1-107-04219-3 - Manifolds, Tensors, and Forms: An Introduction for Mathematicians and Physicists
Paul Renteln
Frontmatter
More information

Contents

Preface

> Q: What's the difference between an argument and a proof? A: An argument will convince a reasonable person, but a proof is needed to convince an unreasonable one.
>
> *Anon.*

> Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so bersetzen sie es in ihre Sprache, und dann ist es alsbald ganz etwas anderes. (Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.)
>
> *Johann Wolfgang von Goethe*

This book offers a concise overview of some of the main topics in differential geometry and topology and is suitable for upper-level undergraduates and beginning graduate students in mathematics and the sciences. It evolved from a set of lecture notes on these topics given to senior-year students in physics based on the marvelous little book by Flanders [25], whose stylistic and substantive imprint can be recognized throughout. The other primary sources used are listed in the references.

By intent the book is akin to a whirlwind tour of many mathematical countries, passing many treasures along the way and only stopping to admire a few in detail. Like any good tour, it supplies all the essentials needed for individual exploration after the tour is over. But, unlike many tours, it also provides language instruction. Not surprisingly, most books on differential geometry are written by mathematicians. This one is written by a mathematically inclined physicist, one who has lived and worked on both sides of the linguistic and formalistic divide that often separates pure and applied mathematics. It is this language barrier that often causes the beginner so much trouble when approaching the subject for the first time. Consequently, the book has been written with a conscious attempt to explain as much as possible from both a "high brow" and a "low brow" viewpoint,[1] particularly in the early chapters.

For many mathematicians, mathematics is the art of avoiding computation. Similarly, physicists will often say that you should never begin a computation unless you know what the answer will be. This may be so, but, more often than not, what happens is that a person works out the answer by ugly computation, and then reworks and publishes the answer in a way that hides all the gory details and makes it seem as though he or she knew the answer all along from pure abstract thought. Still, it is true that there are times when an answer can be obtained much more easily by means of a powerful abstract tool. For this reason, both approaches are given their due here. The result is a compromise between highly theoretical approaches and concrete calculational tools.

This compromise is evident in the use of proofs throughout the book. The one thing that unites mathematicians and scientists is the desire to know, not just *what* is true, but *why* it is true. For this reason, the book contains both proofs and computations. But, in the spirit of the above quotation, arguments sometimes substitute for formal proofs and many long, tedious proofs have been omitted to promote the flow of the exposition. The book therefore risks being not mathematically rigorous enough for some readers and too much so for others, but its virtue is that it is neither encylopedic nor overly pedantic. It is my hope that the presentation will appeal to readers of all backgrounds and interests.

The pace of this work is quick, hitting only the highlights. Although the writing is deliberately terse the tone of the book is for the most part informal, so as to facilitate its use for self-study. Exercises are liberally sprinkled throughout the text and sometimes referred to in later sections; additional problems are placed at the end of most chapters.[2] Although it is not necessary to do all of them, it is certainly advisable to do some; in any case you should read them all, as they provide flesh for the bare bones. After working through this book a student should have acquired all the tools needed to use these concepts in scientific applications. Of course, many topics are omitted and every major topic treated here has many books devoted to it alone. Students wishing to fill in the gaps with more detailed investigations are encouraged to seek out some of the many fine works in the reference list at the end of the book.

[1] The playful epithets are an allusion, of course, to modern humans (abstract thinkers) and Neanderthals (concrete thinkers).

[2] A solutions manual is available to instructors at www.cambridge.org/9781107042193.

The prerequisites for this book are solid first courses in linear algebra, multivariable calculus, and differential equations. Some exposure to point set topology and modern algebra would be nice, but it is not necessary. To help bring students up to speed and to avoid the necessity of looking elsewhere for certain definitions, a mathematics primer is included as Appendix A. Also, the beginning chapter contains all the linear algebra facts employed elsewhere in the book, including a discussion of the correct placement and use of indices.[3] This is followed by a chapter on tensors and multilinear algebra in preparation for the study of tensor analysis and differential forms on smooth manifolds. The de Rham cohomology leads naturally into the topology of smooth manifolds, and from there to a rather brief chapter on the homology of continuous manifolds. The tools introduced there provide a nice way to understand integration on manifolds and, in particular, Stokes' theorem, which is afforded two kinds of treatment. Next we consider vector bundles, connections, and covariant derivatives and then manifolds with metrics. The last chapter offers a very brief introduction to degree theory and some of its uses. This is followed by several appendices providing background material, calculations too long for the main body of the work, or else further applications of the theory.

Originally the book was intended to serve as the basis for a rapid, one-quarter introduction to these topics. But inevitably, as with many such projects, it began to suffer from mission creep, so that covering all the material in ten weeks would probably be a bad idea. Instructors laboring under a short deadline can, of course, simply choose to omit some topics. For example, to get to integration more quickly one could skip Chapter 5 altogether, then discuss only version two of Stokes' theorem. Instructors having the luxury of a semester system should be able to cover everything. Starred sections can be (or perhaps ought to be) skimmed or omitted on a first reading.

This is an expository work drawing freely from many different sources (most of which are listed in the references section), so none of the material harbors any pretense of originality. It is heavily influenced by lectures of Bott and Chern, whose classes I was fortunate enough to take. It also owes a debt to the expository work of many other writers, whose contributions are hereby acknowledged. My sincere apologies to anyone I may have inadvertently missed in my attributions. The manuscript itself was originally typeset using Knuth's astonishingly versatile TEXprogram (and its offspring, LATEX), and the figures were made using Timothy Van Zandt's wonderful graphics tool `pstricks`, enhanced by the three-dimensional drawing packages `pst-3dplot` and `pst-solides3d`.

[3] Stephen Hawking jokes in the introduction to his *Brief History of Time* that the publisher warned him his sales would be halved if he included even one equation. Analogously, sales of this book may be halved by the presence of indices, as many pure mathematicians will do anything to avoid them.

It goes without saying that all writers owe a debt to their teachers. In my case I am fortunate to have learned much from Abhay Ashtekar, Raoul Bott, Shing Shen Chern, Stanley Deser, Doug Eardley, Chris Isham, Karel Kuchař, Robert Lazarsfeld, Rainer Kurt Sachs, Ted Shifrin, Lee Smolin, and Philip Yasskin, who naturally bear all responsibility for any errors contained herein . . . I also owe a special debt to Laurens Gunnarsen for having encouraged me to take Chern's class when we were students together at Berkeley and for other very helpful advice. I am grateful to Rick Wilson and the Mathematics Department at the California Institute of Technology for their kind hospitality over the course of many years and for providing such a stimulating research environment in which to nourish my other life in combinatorics. Special thanks go to Nicholas Gibbons, Lindsay Barnes, and Jessica Murphy at Cambridge University Press, for their support and guidance throughout the course of this project, and to Susan Parkinson, whose remarkable editing skills resulted in many substantial improvements to the book. Most importantly, this work would not exist without the love and affection of my wife, Alison, and our sons David and Michael.