1

Linear algebra

Eighty percent of mathematics is linear algebra.
Raoul Bott

This chapter offers a rapid review of some of the essential concepts of linear algebra that are used in the rest of the book. Even if you had a good course in linear algebra, you are encouraged to skim the chapter to make sure all the concepts and notations are familiar, then revisit it as needed.

1.1 Vector spaces

The standard example of a vector space is $\mathbb{R}^n$, which is the Cartesian product of $\mathbb{R}$ with itself n times: $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$. A vector v in $\mathbb{R}^n$ is an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers with scalar multiplication and vector addition defined as follows†:

$$c(a_1, a_2, \ldots, a_n) := (ca_1, ca_2, \ldots, ca_n) \tag{1.1}$$

and

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n). \tag{1.2}$$

The zero vector is $(0, 0, \ldots, 0)$.

More generally, a **vector space** V over a field $\mathbb{F}$ is a set $\{u, v, w, \ldots\}$ of objects called **vectors**, together with a set $\{a, b, c, \ldots\}$ of elements in $\mathbb{F}$ called **scalars**, that is closed under the taking of linear combinations:

$$u, v \in V \text{ and } a, b \in \mathbb{F} \Rightarrow au + bv \in V, \tag{1.3}$$

and where $0v = 0$ and $1v = v$. (For the full definition, see Appendix A.)

† The notation A := B means that A is defined by B.

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A **subspace** of V is a subset of V that is also a vector space. An **affine subspace** of V is a translate of a subspace of V.[1] The vector space V is the **direct sum** of two subspaces U and W, written $V = U \oplus W$, if $U \cap W = 0$ (the only vector in common is the zero vector) and every vector $v \in V$ can be written uniquely as $v = u + w$ for some $u \in U$ and $w \in W$.

A set $\{v_i\}$ of vectors[2] is **linearly independent** (over the field $\mathbb{F}$) if, for any collection of scalars $\{c_i\} \subset \mathbb{F}$,

$$\sum_i c_i v_i = 0 \qquad \text{implies} \qquad c_i = 0 \text{ for all } i. \tag{1.4}$$

Essentially this means that no member of a set of linearly independent vectors may be expressed as a linear combination of the others.

EXERCISE 1.1 Prove that the vectors $(1, 1)$ and $(2, 1)$ in $\mathbb{R}^2$ are linearly independent over $\mathbb{R}$ whereas the vectors $(1, 1)$ and $(2, 2)$ in $\mathbb{R}^2$ are linearly dependent over $\mathbb{R}$.

A set B of vectors is a **spanning set** for V (or, more simply, **spans** V) if every vector in V can be written as a linear combination of vectors from B. A spanning set of linearly independent vectors is called a **basis** for the vector space. The cardinality of a basis for V is called the **dimension** of the space, written $\dim V$. Vector spaces have many different bases, and they all have the same cardinality. (For the most part we consider only finite dimensional vector spaces.)

Example 1.1 The vector space $\mathbb{R}^n$ is n-dimensional over $\mathbb{R}$. The **standard basis** is the set of n vectors $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)\}$.

Pick a basis $\{e_i\}$ for the vector space V. By definition we may write

$$v = \sum_i v_i e_i \tag{1.5}$$

for any vector $v \in V$, where the v_i are elements of the field $\mathbb{F}$ and are called the **components** of v with respect to the basis $\{e_i\}$. We get a different set of components for the same vector v depending upon the basis we choose.

EXERCISE 1.2 Show that the components of a vector are unique, that is, if $v = \sum_i v_i e_i = \sum_i v'_i e_i$ then $v_i = v'_i$.

EXERCISE 1.3 Show that $\dim(U \oplus W) = \dim U + \dim W$.

[1] By definition, W is a **translate** of U if, for some fixed $v \in V$ with $v \neq 0$, $W = \{u + v : u \in U\}$. An affine subspace is like a subspace without the zero vector.

[2] To avoid cluttering the formulae, the index range will often be left unspecified. In some cases this is because the range is arbitrary, while in other cases it is because the range is obvious.

EXERCISE 1.4 Let W be a subspace of V. Show that we can always **complete a basis** of W to obtain one for V. In other words, if $\dim W = m$ and $\dim V = n$, and if $\{f_1, \dots, f_m\}$ is a basis for W, show there exists a basis for V of the form $\{f_1, \dots, f_m, g_1, \dots, g_{n-m}\}$. Equivalently, show that a basis of a finite-dimensional vector space is just a maximal set of linearly independent vectors.

1.2 Linear maps

Let V and W be vector spaces. A map $T : V \to W$ is **linear** (or a **homomorphism**) if, for $v_1, v_2 \in V$ and $a_1, a_2 \in \mathbb{F}$,

$$T(a_1 v_1 + a_2 v_2) = a_1 T v_1 + a_2 T v_2. \tag{1.6}$$

We will write either $T(v)$ or Tv for the action of the linear map T on a vector v.

EXERCISE 1.5 Show that two linear maps that agree on a basis agree everywhere.

Given a finite subset $U := \{u_1, u_2, \dots\}$ of vectors in V, any map $T : U \to W$ induces a linear map $T : V \to W$ according to the rule $T(\sum_i a_i u_i) := \sum_i a_i T u_i$. The original map is said to have been **extended by linearity**.

The set of all $v \in V$ such that $Tv = 0$ is called the **kernel** (or **null space**) of T, written $\ker T$; $\dim \ker T$ is sometimes called the **nullity** of T. The set of all $w \in W$ for which there exists a $v \in V$ with $Tv = w$ is called the **image** (or **range**) of T, written $\operatorname{im} T$. The **rank** of T, $\operatorname{rk} T$, is defined as $\dim \operatorname{im} T$.

EXERCISE 1.6 Show that $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.

EXERCISE 1.7 Show that T is injective if and only if the kernel of T consists of the zero vector alone.

If T is bijective it is called an **isomorphism**, in which case V and W are said to be **isomorphic**; this is written as $V \cong W$ or, sloppily, $V = W$. Isomorphic vector spaces are not necessarily identical, but they behave as if they were.

Theorem 1.1 *All finite-dimensional vector spaces of the same dimension are isomorphic.*

EXERCISE 1.8 Prove Theorem 1.1.

A linear map from a vector space to itself is called an **endomorphism**, and if it is a bijection it is called an **automorphism**.[3]

[3] Physicists tend to call an endomorphism a **linear operator**.

EXERCISE 1.9 A linear map T is **idempotent** if $T^2 = T$. An idempotent endomorphism $\pi : V \to V$ is called a **projection (operator)**. *Remark:* This is not to be confused with an orthogonal projection, which requires an inner product for its definition.

(a) Show that $V = \operatorname{im} \pi \oplus \ker \pi$.
(b) Suppose W is a subspace of V. Show that there exists a projection operator $\pi : V \to V$ that restricts to the identity map on W. (Note that the projection operator is not unique.) *Hint:* Complete a basis of W so that it becomes a basis of V.

EXERCISE 1.10 Show that if $T : V \to V$ is an automorphism then the inverse map T^{-1} is also linear.

EXERCISE 1.11 Show that the set Aut V of all automorphisms of V is a group. (For information about groups, consult Appendix A.)

1.3 Exact sequences

Suppose that you are given a sequence of vector spaces V_i and linear maps $\varphi_i : V_i \to V_{i+1}$ connecting them, as illustrated below:

$$\cdots \longrightarrow V_{i-1} \xrightarrow{\varphi_{i-1}} V_i \xrightarrow{\varphi_i} V_{i+1} \xrightarrow{\varphi_{i+1}} \cdots .$$

The maps are said to be **exact at** V_i if $\operatorname{im} \varphi_{i-1} = \ker \varphi_i$, i.e., the image of φ_{i-1} equals the kernel of φ_i. The sequence is called an **exact sequence** if the maps are exact at V_i for all i. Exact sequences of vector spaces show up everywhere and satisfy some particularly nice properties, so it worth exploring them a bit.

If V_1, V_2, and V_3 are three vector spaces, and if the sequence

$$0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} 0 \tag{1.7}$$

is exact, it is called a **short exact sequence**. In this diagram "0" represents the zero-dimensional vector space, whose only element is the zero vector. The linear map φ_0 sends 0 to the zero vector of V_1, while φ_3 sends everything in V_3 to the zero vector.

EXERCISE 1.12 Show that the existence of the short exact sequence (1.7) is equivalent to the statement "φ_1 is injective and φ_2 is surjective." In particular, if

$$0 \longrightarrow V \xrightarrow{\varphi} W \longrightarrow 0 \tag{1.8}$$

is exact, V and W must be isomorphic.

It follows that if $T : V \to W$ is surjective then

$$0 \longrightarrow \ker T \xrightarrow{\iota} V \xrightarrow{T} W \longrightarrow 0 \tag{1.9}$$

is a short exact sequence, where ι is the **inclusion map**.[4] By virtue of the above discussion, all short exact sequences are of this form.

Theorem 1.2 *Given the short exact sequence* (1.9)*, there exists a linear map* $S : W \to V$ *such that* $T \circ S = 1$. *We say that the exact sequence* (1.9) **splits**.

Proof Let $\{f_i\}$ be a basis of W. By the surjectivity of T, for each i there exists an $e_i \in V$ such that $T(e_i) = f_i$. Let S be the map $f_i \mapsto e_i$ extended by linearity. The composition of linear maps is linear, so $T \circ S = 1$. (Note that S must therefore be injective.) □

Remark The map S is called a **section** of T.

Theorem 1.3 *Let the short exact sequence* (1.9) *be given, and let* S *be a section of* T. *Then*

$$V = \ker T \oplus S(W).$$

In particular, $\dim V = \dim \ker T + \dim W$.

Proof Let $v \in V$, and define $w := S(T(v))$ and $u := v - w$. Then $T(u) = T(v-w) = T(v) - T(S(T(v))) = 0$, so $v = u + w$ with $u \in \ker T$ and $w \in S(W)$. Now suppose that $x \in \ker T \cap S(W)$. The map S is injective, so $W \cong S(W)$. In particular, there exists a unique $w \in W$ such that $S(w) = x$. But then $w = T(S(w)) = T(x) = 0$, so, by the linearity of S, $x = 0$. Hence $V = \ker T \oplus S(W)$. By Exercise 1.3, $\dim V = \dim \ker T + \dim S(W) = \dim \ker T + \dim W$. □

Remark The conclusion of Theorem 1.3 is commonly referred to as the **rank–nullity theorem**, which states that the rank plus the nullity of any linear map equals the dimension of the domain.

EXERCISE **1.13** Show that if $T : V \to W$ is an injective linear map between spaces of the same dimension then T must be an isomorphism.

EXERCISE **1.14** Let S and T be two endomorphisms of V. Show that $\mathrm{rk}(ST) \le \min\{\mathrm{rk}\, S, \mathrm{rk}\, T\}$.

[4] It is almost silly to give this map a name, because it really doesn't do anything. The idea is that one can view $\ker T$ as a separate vector space as well as a subspace of V, and the inclusion map just sends every element of the first space to the corresponding element of the second.

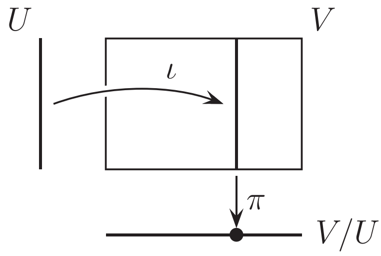

Figure 1.1 The quotient space construction.

1.4 Quotient spaces

Let V be a vector space, and let U be a subspace of V. We define a natural equivalence relation[5] on V by setting $v \sim w$ if $v - w \in U$. The set of equivalence classes of this equivalence relation is denoted V/U and is called the **quotient space of** V **modulo** U. The **(canonical) projection map** is the map $\pi : V \to V/U$ given by $v \to [v]$, where $[v]$ denotes the equivalence class of v in V/U; v is called a **class representative** of $[v]$. The canonical quotient space construction is illustrated in Figure 1.1. The intuition behind the picture is that two distinct points in V that differ by an element of U (or, if you wish, by an element of the image of U under the inclusion map ι) get squashed (or projected) to the same point in the quotient.

Theorem 1.4 *If U is a subspace of the vector space V then V/U carries a natural vector space structure, the projection map is linear, and* $\dim(V/U) = \dim V - \dim U$.

Proof Let $v, w \in V$. Define $[v] + [w] = [v + w]$. We must show that this is well defined, or independent of class representative, meaning that we would get the same answer if we chose different class representatives on the left-hand side. But if $v' \sim v$ and $w' \sim w$ then $v' - v \in U$ and $w' - w \in U$, so $(v' + w') - (v + w) \in U$, which means that $[v' + w'] = [v + w]$. Next, define $a[v] = [av]$ for some scalar a. If $v \sim w$ then $v - w \in U$ so that $a(v - w) \in U$, which means that $[av] = [aw]$. Thus scalar multiplication is also well defined. Hence, V/U equipped with these operations is a vector space. Observe that the zero vector of V/U is just the zero class in V, namely the set of all vectors in U. The definitions imply that $[av + bw] = a[v] + b[w]$, so the projection map is linear. The statement about the dimensions follows from Theorem 1.3 because

$$0 \longrightarrow U \xrightarrow{\iota} V \xrightarrow{\pi} V/U \longrightarrow 0 \tag{1.10}$$

is exact, as π is surjective with $\ker \pi = U$. □

[5] See Appendix A.

Remark We have followed standard usage and called the map $\pi : V \to V/U$ a projection, but technically it is not a projection in the sense of Exercise 1.9 because, as we have defined it, V/U is not a subspace of V. For this reason, although you may be tempted to do so by part (a) of Exercise 1.9, you cannot write $V = U \oplus V/U$. However, by Theorem 1.2, the exact sequence (1.10) splits, so there is a linear injection $S : V/U \to V$. If we define $W = S(V/U)$ then, by Theorem 1.3, $V = U \oplus W$. Conversely, if $V = U \oplus W$ then $W \cong V/U$ (and $U \cong V/W$).

EXERCISE **1.15** Let U be a subspace of V and let W be a subspace of X. Show that any linear map $\varphi : V \to X$ with $\varphi(U) \subseteq W$ induces a natural linear map $\widetilde{\varphi} : V/U \to X/W$. *Hint:* Set $\widetilde{\varphi}([v]) = [\varphi(v)]$. Show this is well defined.

1.5 Matrix representations

Let V be n-dimensional. Given an endomorphism $T : V \to V$ together with a basis $\{e_i\}$ of V we can construct an $n \times n$ matrix whose entries T_{ij} are given by[6]

$$Te_j = \sum_i e_i T_{ij}. \tag{1.11}$$

One writes (T_{ij}) or $\boldsymbol{T}$ to indicate the matrix whose entries are T_{ij}. The map $T \to \boldsymbol{T}$ is called a **representation** of T (in the basis $\{e_i\}$). A different choice of basis leads to a different matrix, but they both represent the same endomorphism.

Let $v = \sum_i v_i e_i \in V$. Then

$$\begin{aligned} v' := Tv = \sum_j v_j Te_j &= \sum_{ij} v_j e_i T_{ij} \\ &= \sum_i \left(\sum_j T_{ij} v_j \right) e_i = \sum_i v'_i e_i, \end{aligned}$$

so the components of v' are related to those of v according to the rule

$$v'_i = \sum_j T_{ij} v_j. \tag{1.12}$$

EXERCISE **1.16** Let S and T be two endomorphisms of V. Show that if $S \to \boldsymbol{S}$ and $T \to \boldsymbol{T}$ then $ST \to \boldsymbol{ST}$, where $ST := S \circ T$, the composition of S and T, and matrix multiplication is defined by

$$(\boldsymbol{ST})_{ij} = \sum_k S_{ik} T_{kj}. \tag{1.13}$$

(This shows why matrix multiplication is defined in the way it is.)

[6] Note carefully the placement of the indices.

EXERCISE 1.17 Let $T : V \to V$ be an automorphism represented by $\boldsymbol{T}$. Show that $T^{-1} \to \boldsymbol{T}^{-1}$, where $\boldsymbol{T}^{-1}$ denotes the inverse matrix.

The **row rank** (respectively, **column rank**) of a matrix $\boldsymbol{T}$ is the maximum number of linearly independent rows (respectively, columns), when they are considered as vectors in $\mathbb{R}^n$. The row rank and column rank are always equal, and they equal the **rank** $\operatorname{rk} \boldsymbol{T}$ of $\boldsymbol{T}$. If $\operatorname{rk} \boldsymbol{T}$ equals n then we say that $\boldsymbol{T}$ has **maximal rank**; otherwise it is said to be **rank deficient**.

EXERCISE 1.18 Show that the rank of the endomorphism T equals the rank of the matrix $\boldsymbol{T}$ representing it in any basis.

1.6 The dual space

A **linear functional** on V is a linear map $f : V \to \mathbb{F}$. The set V^* of all linear functionals on V is called the **dual space** (of V), and is often denoted $\operatorname{Hom}(V, \mathbb{R})$. If f is a linear functional and a is a scalar, af is another linear functional, defined by $(af)(v) = af(v)$ (pointwise multiplication). Also, if f and g are two linear functionals then we can obtain a third linear functional $f + g$ by $(f + g)(v) = f(v) + g(v)$ (pointwise addition). These two operations turn V^* into a vector space, and when one speaks of the dual space one always has this vector space structure in mind.

It is customary to write $\langle v, f \rangle$ or $\langle f, v \rangle$ to denote $f(v)$.[7] When written this way it is called the **natural pairing** or **dual pairing** between V and V^*. Elements of V^* are often called **covectors**.

If $\{e_i\}$ is a basis of V, there is a canonical **dual basis** or **cobasis** $\{\theta_j\}$ of V^*, defined by $\langle e_i, \theta_j \rangle = \delta_{ij}$, where δ_{ij} is the **Kronecker delta**:

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \text{, and} \\ 0 & \text{otherwise.} \end{cases} \tag{1.14}$$

Any element $f \in V^*$ can be expanded in terms of the dual basis as

$$f = \sum_i f_i \theta_i, \tag{1.15}$$

where $f_i \in \mathbb{F}$. The scalars f_i are called the **components** of f with respect to the basis $\{\theta_i\}$.

EXERCISE 1.19 Show that $\{\theta_j\}$ is indeed a basis for V^*.

7 Note the angular shape of the brackets.

It follows from Exercise 1.19 that $\dim V^* = \dim V$. Because they have the same dimension, V and V^* are isomorphic, but not in any natural way.[8] On the other hand, V and V^{**} (the **double dual** of V) are always isomorphic via the natural map $v \mapsto (f \mapsto f(v))$.[9]

EXERCISE 1.20 Show that the dual pairing is **nondegenerate** in the following sense. If $\langle f, v\rangle = 0$ for all v then $f = 0$ and if $\langle f, v\rangle = 0$ for all f then $v = 0$.

EXERCISE 1.21 Let W be a subspace of V. The **annihilator** of W, denoted $\operatorname{Ann} W$, is the set of all linear functionals that map every element of W to zero:

$$\operatorname{Ann} W := \{\theta \in V^* : \theta(w) = 0, \text{ for all } w \in W\}. \tag{1.16}$$

Show that $\operatorname{Ann} W$ is a subspace of V^* and that every subspace of V^* is $\operatorname{Ann} W$ for some W. In the process verify that $\dim V = \dim W + \dim \operatorname{Ann} W$. *Hint:* For the second part, let U^* be a subspace of V^* and define

$$W := \{v \in V : f(v) = 0, \text{ for all } f \in U^*\}.$$

Use this to show that $U^* \subseteq \operatorname{Ann} W$. Equality then follows by a dimension argument.

EXERCISE 1.22 Let W be a subspace of V. Then $(V/W)^* \cong V^*/W^*$ just by dimension counting, even though there is no natural isomorphism (because there is no natural isomorphism between a vector space and its dual). But there is an interesting connection to annihilators that gives an indirect relation between the two spaces.

(a) Show that $(V/W)^* \cong \operatorname{Ann} W$. *Hint:* Define a map $\varphi : \operatorname{Ann} W \to (V/W)^*$ by $f \mapsto \varphi(f)$, where $\varphi(f)([v]) := f(v)$, and extend by linearity. Show that φ is well defined and an isomorphism.
(b) Show that $V^*/\operatorname{Ann} W \cong W^*$. *Hint:* Let $\pi : V \to W$ be any projection onto W, and define $\pi^* : V^*/\operatorname{Ann} W \to W^*$ by $[f] \mapsto f \circ \pi$, so $\pi^*([f])(v) = f(\pi(v))$. Again, show that π^* is well defined and an isomorphism.

1.7 Change of basis

Let $\{e_i\}$ and $\{e'_i\}$ be two bases of V. Each new (primed) basis vector can be written as a linear combination of the old (unprimed) basis vectors, and by convention we write

$$e'_j = \sum_i e_i A_{ij} \tag{1.17}$$

[8] The situation is different if V is an inner product space. See Section 1.10.
[9] When V is infinite dimensional, the situation is more subtle. In that case the statements $V \cong V^*$ and $V \cong V^{**}$ are both generally false. A notable exception occurs when V is a Hilbert space and V^* is defined as the continuous linear functionals on V. See e.g. [74].

for some nonsingular matrix $\boldsymbol{A} = (A_{ij})$, called the **change of basis matrix**.[10] By definition, a change of basis leaves all the vectors in V untouched – it merely changes their *description*. Thus, if $v = \sum_i v_i e_i$ is the expansion of v relative to the old basis and $v = \sum_i v'_i e'_i$ is the expansion of v relative to the new basis then

$$\sum_j v'_j e'_j = \sum_{ij} v'_j e_i A_{ij} = \sum_i \left(\sum_j A_{ij} v'_j \right) e_i = \sum_i v_i e_i.$$

Hence,

$$v_i = \sum_j A_{ij} v'_j \qquad \text{or} \qquad v'_i = \sum_j (A^{-1})_{ij} v_j. \tag{1.18}$$

Note that the basis vectors and the components of vectors transform differently under a change of basis.

A change of basis on V induces a change of basis on V^* owing to the requirement that the natural dual pairing be preserved. Thus, if $\{\theta_i\}$ and $\{\theta'_i\}$ are the dual bases corresponding to $\{e_i\}$ and $\{e'_i\}$, respectively, then we demand that

$$\langle e'_i, \theta'_j \rangle = \langle e_i, \theta_j \rangle = \delta_{ij}. \tag{1.19}$$

Writing

$$\theta'_j = \sum_i \theta_i B_{ij} \tag{1.20}$$

for some matrix $\boldsymbol{B} = (B_{ij})$ and using (1.19) gives[11]

$$\begin{aligned} \delta_{ij} = \langle e'_i, \theta'_j \rangle &= \sum_{k\ell} \langle e_k A_{ki}, \theta_\ell B_{\ell j} \rangle = \sum_{k\ell} A_{ki} B_{\ell j} \langle e_k, \theta_\ell \rangle \\ &= \sum_{k\ell} A_{ki} B_{\ell j} \delta_{k\ell} = \sum_k A_{ki} B_{kj}. \end{aligned} \tag{1.21}$$

Writing $\boldsymbol{A}^T$ for the **transpose** matrix, whose entries are $(\boldsymbol{A}^T)_{ij} := A_{ji}$, (1.21) can be written compactly as $\boldsymbol{A}^T \boldsymbol{B} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Equivalently, we have $\boldsymbol{B} = (\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$, the so-called **contragredient matrix** of $\boldsymbol{A}$.

[10] Equation (1.17) looks a lot like (1.11), so much so that one can define a linear map $A : V \to V$, called the **change of basis map**, given by $e_i \mapsto e'_i$ and extended by linearity. This map can be a bit confusing, though, for the following reason. In general, a map $T : V \to W$ is represented by a matrix (T_{ij}) obtained from $Te_i = \sum_j T_{ji} f_j$, where $\{e_i\}$ is a basis for V and $\{f_i\}$ is a basis for W. Viewing A as a map from V equipped with the basis $\{e_i\}$ to V equipped with the basis $\{e'_i\}$, the map A is represented by the identity matrix even though it is not the identity map. Instead, the change of basis matrix is what we obtain if we use the same basis $\{e_i\}$ for both copies of V. To avoid these sorts of confusion we shall speak only of the change of basis matrix when discussing basis changes.

[11] Sometimes the last step in (1.21) gives beginners a little trouble. We are using the substitution property of the Kronecker delta. For example, $\sum_\ell v_\ell \delta_{\ell m} = v_m$, because the Kronecker delta vanishes unless $\ell = m$. To be really explicit, if, say, $m = 2$ and the indices run from 1 to 3, then $\sum_\ell v_\ell \delta_{\ell 2} = v_1 \delta_{12} + v_2 \delta_{22} + v_3 \delta_{32} = v_2$, because $\delta_{12} = \delta_{32} = 0$ and $\delta_{22} = 1$.