Analysis of Boolean Functions

Boolean functions are perhaps the most basic objects of study in theoretical computer science. They also arise in other areas of mathematics, including combinatorics, statistical physics, and mathematical social choice. The field of analysis of Boolean functions seeks to understand them via their Fourier transform and other analytic methods. This text gives a thorough overview of the field, beginning with the most basic definitions and proceeding to advanced topics such as hypercontractivity and isoperimetry. Each chapter includes a "highlight application" such as Arrow's theorem from economics, the Goldreich-Levin algorithm from cryptography/learning theory, Håstad's NP-hardness of approximation results, and "sharp threshold" theorems for random graph properties. The book includes nearly 500 exercises and can be used as the basis of a one-semester graduate course. It should appeal to advanced undergraduates, graduate students, and researchers in computer science theory and related mathematical fields.

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Additional resources for this publication at http://analysisofbooleanfunctions.org

To Zeynep,
for her unending support and encouragement.

Contents

Preface

The subject of this textbook is the *analysis of Boolean functions*. Roughly speaking, this refers to studying Boolean functions $f : \{0, 1\}^n \to \{0, 1\}$ via their Fourier expansion and other analytic means. Boolean functions are perhaps the most basic object of study in theoretical computer science, and Fourier analysis has become an indispensable tool in the field. The topic has also played a key role in several other areas of mathematics, from combinatorics, random graph theory, and statistical physics, to Gaussian geometry, metric/Banach spaces, and social choice theory.

The intent of this book is both to develop the foundations of the field and to give a wide (though far from exhaustive) overview of its applications. Each chapter ends with a "highlight" showing the power of analysis of Boolean functions in different subject areas: property testing, social choice, cryptography, circuit complexity, learning theory, pseudorandomness, hardness of approximation, concrete complexity, and random graph theory.

The book can be used as a reference for working researchers or as the basis of a one-semester graduate-level course. The author has twice taught such a course at Carnegie Mellon University, attended mainly by graduate students in computer science and mathematics but also by advanced undergraduates, postdocs, and researchers in adjacent fields. In both years most of Chapters 1–5 and 7 were covered, along with parts of Chapters 6, 8, 9, and 11, and some additional material on additive combinatorics. Nearly 500 exercises are provided at the ends of the book's chapters.

Additional material related to the book can be found at its website:

http://analysisofbooleanfunctions.org

This includes complete lecture notes from the author's 2007 course, complete lecture videos from the author's 2012 course, blog updates related to analysis of Boolean functions, an electronic draft of the book, and errata. The author would like to encourage readers to post any typos, bugs, clarification requests, and suggestions to this website.

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I worked on this book while visiting the Institute for Advanced Study in 2010–2011 (supported by the Von Neumann Fellowship and in part by NSF grants DMS-0835373 and CCF-0832797); I'm very grateful to them for having me and for the wonderful working environment they provided. The remainder of the work on this book was done at Carnegie Mellon; I'm of course very thankful to my colleagues there and to the Department of Computer Science. "Reasonable" random variables were named after the department's "Reasonable Person Principle." I was also supported in this book-writing endeavor by the National Science Foundation, specifically grants CCF-0747250 and CCF-1116594. As usual: "This material is based upon work supported by the National Science Foundation under grant numbers listed above. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation (NSF)."

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Ryan O'Donnell
Pittsburgh
October 2013

List of Notation

$\circ$	entry-wise multiplication of vectors
∇	the gradient: $\nabla f(x) = (\mathrm{D}_1 f(x), \dots, \mathrm{D}_n f(x))$
$\neg$	logical NOT
$\ni$	$S \ni i$ is equivalent to $i \in S$
$\oplus$	logical XOR (exclusive-or)
$\hat{\lVert} f \hat{\rVert}_p$	$(\sum_{\gamma \in \widehat{\mathbb{F}_2^n}} \lvert\widehat{f}(\gamma)\rvert^p)^{1/p}$
$\triangle$	symmetric difference of sets; i.e., $S \triangle T = \{i : i \text{ is in exactly one of } S, T\}$
$\vee$	logical OR
$\wedge$	logical AND
$*$	the convolution operator
$[z^k]F(z)$	coefficient on z^k in the power series $F(z)$
1_A	0-1 indicator function for A
$\mathbf{1}_B$	0-1 indicator random variable for event B
2^A	the set of all subsets of A
$\#\alpha$	if α is a multi-index, denotes the number of nonzero components of α
$\lvert\alpha\rvert$	if α is a multi-index, denotes $\sum_i \alpha_i$
AND_n	the logical AND function on n bits: False unless all inputs are True
$A^\perp$	$\{\gamma : \gamma \cdot x = 0 \text{ for all } x \in A\}$
$\mathrm{Aut}(f)$	the group of automorphisms of Boolean function f
$\mathrm{BitsToGaussians}_M^d$	on input the bit matrix $x \in \{-1,1\}^{d \times M}$ has output $z \in \mathbb{R}^d$ equal to $\frac{1}{\sqrt{M}}$ times the column-wise sum of x; if d is omitted it's taken to be 1
$\mathbb{C}$	the complex numbers
$\chi(b)$	when $b \in \mathbb{F}_2^n$, denotes $(-1)^b \in \mathbb{R}$

$\chi_S(x)$	when $x \in \mathbb{R}^n$, denotes $\prod_{i \in S} x_i$, where $S \subseteq [n]$; when $x \in \mathbb{F}_2^n$, denotes $(-1)^{\sum_{i \in S} x_i}$
codim H	for a subspace $H \leq \mathbb{F}^n$, denotes $n - \dim H$
$\mathbf{Cov}[f, g]$	the covariance of f and g, $\mathbf{Cov}[f] = \mathbf{E}[fg] - \mathbf{E}[f]\,\mathbf{E}[g]$
D_i	the ith discrete derivative: $\mathrm{D}_i f(x) = \frac{f(x^{(i \mapsto 1)}) - f(x^{(i \mapsto -1)})}{2}$
$d_{\chi^2}(\varphi, 1)$	chi-squared distance of the distribution with density φ from the uniform distribution
$\deg(f)$	the degree of f; the least k such that f is a real linear combination of k-juntas
$\deg_{\mathbb{F}_2}(f)$	for Boolean-valued f, the degree of its $\mathbb{F}_2$-polynomial representation
$\Delta(x, y)$	the Hamming distance, $\#\{i : x_i \neq y_i\}$
$\Delta^{(\pi)}(f)$	the expected number of queries made by the best decision tree computing f when the input bits are chosen from the distribution π
$\delta^{(\pi)}(f)$	the revealment of f; i.e., $\min\{\max_i \delta_i^{(\pi)}(\mathcal{T}) : \mathcal{T} \text{ computes } f\}$
$\Delta^{(\pi)}(\mathcal{T})$	the expected number of queries made by randomized decision tree $\mathcal{T}$ when the input bits are chosen from the distribution π
$\delta_i^{(\pi)}(\mathcal{T})$	the probability randomized decision tree $\mathcal{T}$ queries coordinate i when the input bits are chosen from the distribution π
$\Delta_y f$	for $f : \mathbb{F}_2^n \to \mathbb{F}_2$, the function $\mathbb{F}_2^n \to \mathbb{F}_2$ defined by $\Delta_y f(x) = f(x + y) - f(x)$
$\mathrm{dist}(g, h)$	the relative Hamming distance; i.e., the fraction of inputs on which g and h disagree
$\mathrm{DNF}_{\mathrm{size}}(f)$	least possible size of a DNF formula computing f
$\mathrm{DNF}_{\mathrm{width}}(f)$	least possible width of a DNF formula computing f
$\mathrm{DT}(f)$	least possible depth of a decision tree computing f
$\mathrm{DT}_{\mathrm{size}}(f)$	least possible size of a decision tree computing f
$d_{\mathrm{TV}}(\varphi, \psi)$	total variation distance between the distributions with densities φ, ψ
E_i	the ith expectation operator: $\mathrm{E}_i f(x) = \mathbf{E}_{\boldsymbol{x}_i}[f(x_1, \dots, x_{i-1}, \boldsymbol{x}_i, x_{i+1}, \dots, x_n))]$
E_I	the expectation over coordinates I operator
$\mathbf{Ent}[f]$	for a nonnegative function on a probability space, denotes $\mathbf{E}[f \ln f] - \mathbf{E}[f] \ln \mathbf{E}[f]$
$\mathbf{E}_{\pi_p}[\cdot]$	an abbreviation for $\mathbf{E}_{\boldsymbol{x} \sim \pi_p^{\otimes n}}[\cdot]$

$f \oplus g$	if $f : \{-1,1\}^m \to \{-1,1\}$ and $g : \{-1,1\}^n \to \{-1,1\}$, denotes the function $h : \{-1,1\}^{m+n} \to \{-1,1\}$ defined by $h(x, y) = f(x)g(y)$
$f \otimes g$	if $f : \{-1,1\}^m \to \{-1,1\}$ and $g : \{-1,1\}^n \to \{-1,1\}$, denotes the function $h : \{-1,1\}^{mn} \to \{-1,1\}$ defined by $h(x^{(1)}, \dots, x^{(m)}) = f(g(x^{(1)}), \dots, g(x^{(m)}))$
$f^{\otimes d}$	if $f : \{-1,1\}^n \to \{-1,1\}$, then $f^{\otimes d} : \{-1,1\}^{n^d} \to \{-1,1\}$ is defined inductively by $f^{\otimes 1} = f$, $f^{\otimes(d+1)} = f \otimes f^{\otimes d}$
f^{*n}	the n-fold convolution, $f * f * \cdots * f$
$f^{\dagger}$	the Boolean dual defined by $f^{\dagger}(x) = -f(-x)$
f^{+z}	if $f : \mathbb{F}_2^n \to \mathbb{R}$, $z \in \mathbb{F}_2^n$, denotes the function $f^{+z}(x) = f(x + z)$
f_H^{+z}	denotes $(f^{+z})_H$
$\mathbb{F}_2$	the finite field of size 2
$\widehat{\mathbb{F}_2^n}$	the group (vector space) indexing the Fourier characters of functions $f : \mathbb{F}_2^n \to \mathbb{R}$
f^{even}	the even part of f, $(f(x) + f(-x))/2$
$\langle f, g \rangle$	$\mathbf{E}_{\boldsymbol{x}}[f(\boldsymbol{x})g(\boldsymbol{x})]$
f_H	if $f : \mathbb{F}_2^n \to \mathbb{R}$, $H \le \mathbb{F}_2^n$, denotes the restriction of f to H
$\widehat{f}(i)$	shorthand for $\widehat{f}(\{i\})$ when $i \in \mathbb{N}$
$f^{\subseteq J}$	the function (depending only on the J coordinates) defined by $f^{\subseteq J}(x) = \mathbf{E}_{\boldsymbol{x}'_{\overline{J}}}[f(x_J, \boldsymbol{x}'_{\overline{J}})]$; in particular, it's $\sum_{S \subseteq J} \widehat{f}(S)\chi_S$ when $f : \{-1,1\}^n \to \mathbb{R}$
$f_{\mid z}$	if $f : \Omega^n \to \mathbb{R}$, $J \subseteq [n]$, and $z \in \Omega^{\overline{J}}$, denotes the restriction of f given by fixing the coordinates in $\overline{J}$ to z
$f_{J\mid z}$	if $f : \Omega^n \to \mathbb{R}$, $J \subseteq [n]$, and $z \in \Omega^{\overline{J}}$, denotes the restriction of f given by fixing the coordinates in $\overline{J}$ to z
$f^{=k}$	$\sum_{\lvert S\rvert = k} \widehat{f}(S)\,\chi_S$
$f^{\le k}$	$\sum_{\lvert S\rvert \le k} \widehat{f}(S)\,\chi_S$
f^{odd}	the odd part of f, $(f(x) - f(-x))/2$
$\mathbb{F}_{p^\ell}$	for p prime and $\ell \in \mathbb{N}^+$, denotes the finite field of p^ℓ elements
$\widehat{f}(S)$	the Fourier coefficient of f on character χ_S
$\mathrm{F}_{S\mid\overline{J}}f(z)$	for $S \subseteq J \subseteq [n]$, denotes $\widehat{f_{J\mid z}}(S)$
$\widetilde{f}$	the randomization/symmetrization of f, defined by $\widetilde{f}(r, x) = \sum_S \boldsymbol{r}^S f^{=S}(\boldsymbol{x})$
$\gamma^{+}(\partial A)$	the Gaussian Minkowski content of ∂A
$\mathcal{G}(v, p)$	the Erdős–Rényi random graph distribution, $\pi_p^{\otimes\binom{v}{2}}$

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More information

h_j	the jth (normalized) Hermite polynomial, $h_j = \frac{1}{\sqrt{j!}} H_j$
h_α	for $\alpha \in \mathbb{N}^n$ a multi-index, the n-variate (normalized) Hermite polynomial $h_\alpha(z) = \prod_{j=1}^n h_{\alpha_j}(z_j)$
H_j	the jth probabilists' Hermite polynomial, defined by $\exp(tz - \frac{1}{2}t^2) = \sum_{j=0}^\infty \frac{1}{j!} H_j(z) t^j$
$\mathbf{Inf}_i[f]$	the influence of coordinate i on f
$\mathbf{Inf}_i^{(\rho)}[f]$	the ρ-stable influence, $\mathbf{Stab}_\rho[\mathrm{D}_i f]$
$\widetilde{\mathbf{Inf}}_J[f]$	the coalitional influence of $J \subseteq [n]$ on $f : \{-1,1\}^n \to \{-1,1\}$, namely $\mathbf{Pr}_{z \sim \{-1,1\}^{\overline{J}}}[f_{J\mid z}$ is not constant$]$
$\widetilde{\mathbf{Inf}}_J^b[f]$	equals $\mathbf{Pr}_{z \sim \{-1,1\}^{\overline{J}}}[f_{J\mid z} \not\equiv -b] - \mathbf{Pr}[f = b]$, for $b \in \{-1,1\}$
$\overline{J}$	if $J \subseteq [n]$, denotes $[n] \setminus J$
$L^2(\{-1,1\}^n)$	denotes $L^2(\{-1,1\}^n, \pi_{1/2}^{\otimes n})$
$L^2(G^n)$	if G is a finite abelian group, denotes the complex inner product space of functions $G^n \to \mathbb{R}$ with inner product $\langle f, g\rangle = \mathbf{E}_{\boldsymbol{x} \sim G^n}[f(\boldsymbol{x})\overline{g(\boldsymbol{x})}]$
$L^2(\Omega, \pi)$	the inner product space of (square-integrable) functions $\Omega \to \mathbb{R}$ with inner product $\langle f, g\rangle = \mathbf{E}_{\boldsymbol{x} \sim \pi}[f(\boldsymbol{x})g(\boldsymbol{x})]$
$\Lambda_\rho(\alpha, \beta)$	$\mathbf{Pr}[z_1 \le t, z_2 \le t']$, where z_1, z_2 are standard Gaussians with correlation $\mathbf{E}[z_1 z_2] = \rho$, and $t = \Phi^{-1}(\alpha)$, $t' = \Phi^{-1}(\beta)$
$\Lambda_\rho(\alpha)$	denotes $\Lambda_\rho(\alpha, \alpha)$
$\mathrm{L}f$	the Laplacian operator applied to the Boolean function f, defined by $\mathrm{L}f = \sum_{i=1}^n \mathrm{L}_i f$ (or, the Ornstein–Uhlenbeck operator if f is a function on Gaussian space)
L_i	the ith coordinate Laplacian operator: $\mathrm{L}_i f = f - \mathrm{E}_i f$
$\ln x$	$\log_e x$
$\log x$	$\log_2 x$
Maj_n	the majority function on n bits
$\mathbf{MaxInf}[f]$	$\max_i\{\mathbf{Inf}_i[f]\}$
$[n]$	$\{1, 2, 3, \dots, n\}$
$\mathbb{N}$	$\{0, 1, 2, 3, \dots\}$
$\mathbb{N}^+$	$\{1, 2, 3, \dots\}$
$\mathbb{N}_{<m}$	$\{0, 1, \dots, m-1\}$
$N_\rho(x)$	when $x \in \{-1,1\}^n$, denotes the probability distribution generating a string ρ-correlated to x
$N_\rho(z)$	when $z \in \mathbb{R}^n$, denotes the probability distribution of $\rho z + \sqrt{1-\rho^2}\,\boldsymbol{g}$ where $\boldsymbol{g} \sim \mathrm{N}(0,1)^n$
$\mathbf{NS}_\delta[f]$	the noise sensitivity of f at δ; i.e., $\frac{1}{2} - \frac{1}{2}\mathbf{Stab}_{1-2\delta}[f]$

$\mathrm{N}(0,1)$	the standard Gaussian distribution
$\mathrm{N}(0,1)^d$	the distribution of d independent standard Gaussians; i.e., $\mathrm{N}(0, I_{d\times d})$
$\mathrm{N}(\mu,\Sigma)$	for $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$ positive semidefinite, the d-variate Gaussian distribution with mean μ and covariance matrix Σ
OR_n	the logical OR function on n bits: True unless all inputs are False
ϕ	the standard Gaussian pdf, $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$
Φ	the standard Gaussian cdf, $\Phi(t) = \int_{-\infty}^{t} \phi(z)\,dz$
$\overline{\Phi}$	the standard Gaussian complementary cdf, $\overline{\Phi}(t) = \int_t^{\infty} \phi(z)\,dz$
φ_A	the density function for the uniform probability distribution on A; i.e., $1_A / \mathbf{E}[1_A]$
ϕ_α	given functions $\phi_0, \dots, \phi_{m-1}$ and a multi-index α, denotes $\prod_{i=1}^n \phi_{\alpha_i}$
$\pi^{\otimes n}$	if π is a probability distribution on Ω, denotes the associated product probability distribution on Ω^n
$\pi_{1/2}$	the uniform distribution on $\{-1, 1\}$
π_p	the "p-biased" distribution on bits: $\pi_p(-1) = p$, $\pi_p(1) = 1 - p$
$\mathbf{Pr}_{\pi_p}[\cdot]$	an abbreviation for $\mathbf{Pr}_{\boldsymbol{x}\sim\pi_p^{\otimes n}}[\cdot]$
$\mathbb{R}$	the real numbers
$\mathbb{R}^{\geq 0}$	the nonnegative real numbers
$\mathrm{RDT}(f)$	the zero-error randomized decision tree complexity of f
$\mathbf{RS}_A(\delta)$	the rotation sensitivity of A at δ; i.e., $\mathbf{Pr}[1_A(\boldsymbol{z}) \neq 1_A(\boldsymbol{z}')]$ for a $\cos\delta$-correlated pair $(\boldsymbol{z}, \boldsymbol{z}')$
$\mathrm{sens}_f(x)$	the number of pivotal coordinates for f at x
$\mathrm{sgn}(t)$	$+1$ if $t \geq 0$, -1 if $t < 0$
S_n	the symmetric group on $[n]$
$\mathrm{sparsity}(f)$	$\mathbf{Pr}_{\boldsymbol{x}}[f(\boldsymbol{x}) \neq 0]$
$\mathrm{sparsity}(\widehat{f})$	$\lvert\mathrm{supp}(\widehat{f})\rvert$
$\mathbf{Stab}_\rho[f]$	the noise stability of f at ρ: $\mathbf{E}[f(\boldsymbol{x})f(\boldsymbol{y})]$ where $\boldsymbol{x}, \boldsymbol{y}$ are a ρ-correlated pair
$\mathrm{supp}(\alpha)$	if α is a multi-index, denotes $\{i : \alpha_i \neq 0\}$
$\mathrm{supp}(f)$	if f is a function, denotes the set of inputs where f is nonzero
T_ρ	the noise operator: $\mathrm{T}_\rho f(x) = \mathbf{E}_{\boldsymbol{y}\sim N_\rho(x)}[f(\boldsymbol{y})]$
T_ρ^i	the operator defined by $\mathrm{T}_\rho^i f(x) = \rho f + (1-\rho)\mathrm{E}_i f$

T_r	for $r \in \mathbb{R}^n$, denotes the operator defined by $\mathrm{T}^1_{r_1}\mathrm{T}^2_{r_2}\cdots\mathrm{T}^n_{r_n}$
$\mathscr{U}$	the Gaussian isoperimetric function, $\mathscr{U} = \phi \circ \Phi^{-1}$
U_ρ	the Gaussian noise operator: $\mathrm{U}_\rho f(z) = \mathbf{E}_{z' \sim N_\rho(z)}[f(z')]$
$\mathbf{Var}[f]$	the variance of f, $\mathbf{Var}[f] = \mathbf{E}[f^2] - \mathbf{E}[f]^2$
Var_i	the operator defined by $\mathrm{Var}_i f(x) = \mathbf{Var}_{\boldsymbol{x}_i}[f(x_1, \ldots, x_{i-1}, \boldsymbol{x}_i, x_{i+1}, \ldots, x_n))]$
$\mathrm{vol}_\gamma(A)$	$\mathbf{Pr}_{z \sim \mathrm{N}(0,1)^n}[z \in A]$, the Gaussian volume of A
$\mathbf{W}^k[f]$	the Fourier weight of f at degree k
$\mathbf{W}^{>k}[f]$	the Fourier weight of f at degrees above k
$x^{(i \mapsto b)}$	the string $(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$
$x^{\oplus i}$	$(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$
$\boldsymbol{x} \sim \varphi$	the random variable $\boldsymbol{x}$ is chosen from the probability distribution with density φ
x^S	$\prod_{i \in S} x_i$, with the convention $x^\emptyset = 1$
$\boldsymbol{x} \sim A$	the random variable $\boldsymbol{x}$ is chosen uniformly from the set A
$\boldsymbol{x} \sim \{-1,1\}^n$	the random variable $\boldsymbol{x}$ is chosen uniformly from $\{-1,1\}^n$
(y, z)	if $J \subseteq [n]$, $y \in \{-1,1\}^J$, $z \in \{-1,1\}^{\overline{J}}$, denotes the natural composite string in $\{-1,1\}^n$
$\mathbb{Z}$	the additive group of integers modulo m
$\widehat{\mathbb{Z}^n_m}$	the group indexing the Fourier characters of functions $f : \mathbb{Z}^n_m \to \mathbb{C}$