

Elements of Logical Reasoning

Some of our earliest experiences of the conclusive force of an argument come from school mathematics: faced with a mathematical proof, we cannot deny the conclusion once the premisses have been accepted. Behind such arguments lies a more general pattern of ‘demonstrative arguments’ that is studied in the science of logic. Logical reasoning is applied at all levels, from everyday life to advanced sciences, and a remarkable level of complexity is achieved in everyday logical reasoning, even if the principles behind it remain intuitive. Jan von Plato provides an accessible but rigorous introduction to an important aspect of contemporary logic: its deductive machinery. He shows that when the forms of logical reasoning are analysed, it turns out that a limited set of first principles can represent any logical argument. His book will be valuable for students of logic, mathematics, and computer science.

JAN VON PLATO is Professor of Philosophy at the University of Helsinki. He is the author of *Creating Modern Probability* (Cambridge, 1994) and, with Sara Negri, *Structural Proof Theory* (Cambridge, 2001) and *Proof Analysis* (Cambridge, 2011).

Cambridge University Press
978-1-107-03659-8 - Elements of Logical Reasoning
Jan Von Plato
Frontmatter
[More information](#)

Cambridge University Press
978-1-107-03659-8 - Elements of Logical Reasoning
Jan Von Plato
Frontmatter
[More information](#)

Elements of Logical Reasoning

JAN VON PLATO



Cambridge University Press
978-1-107-03659-8 - Elements of Logical Reasoning
Jan Von Plato
Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is a part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107036598

© Jan von Plato 2013

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2013

Printed in the United Kingdom by MPG Printgroup Ltd, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Von Plato, Jan, author.

Elements of logical reasoning / Jan von Plato.

pages cm

Includes bibliographical references and index.

ISBN 978-1-107-03659-8 (hardback) – ISBN 978-1-107-61077-4 (paperback)

1. Logic, Symbolic and mathematical. 2. Reasoning. I. Title.

QA9.V66 2013

511.3 – dc23 2013039013

ISBN 978-1-107-03659-8 Hardback

ISBN 978-1-107-61077-4 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Contents

Preface [page ix]

PART I FIRST STEPS IN LOGICAL REASONING

- 1 Starting points [3]
 - 1.1 Origins [3]
 - 1.2 Demonstrative arguments [4]
 - 1.3 Propositions and assertions [7]
 - 1.4 The connectives [8]
 - 1.5 Grammatical variation, unique readability [10]
 - 1.6 A grammar for propositional logic [12]
 - 1.7 Idealization [14]
- 2 Rules of proof [15]
 - 2.1 Steps in proofs [15]
 - 2.2 Negation [18]
 - 2.3 Natural deduction in a linear form [19]
 - 2.4 The notion of a derivation [21]
 - 2.5 How to construct derivations? [23]
 - 2.6 Schematic formulas, schematic derivations [25]
 - 2.7 The structure of derivations [27]
 - Notes and exercises to Chapter 2 [29]
- 3 Natural deduction [31]
 - 3.1 From linear derivations to derivation trees [31]
 - 3.2 Gentzen's rules of natural deduction [34]
 - 3.3 Derivations with cases [38]
 - 3.4 A final modification [42]
 - 3.5 The role of falsity and negation [46]
 - 3.6 Axiomatic logic [49]
 - 3.7 Proofs of unprovability [58]
 - 3.8 Meaning explanations [61]
 - Notes and exercises to Chapter 3 [62]
- 4 Proof search [64]
 - 4.1 Naturally growing trees [64]
 - 4.2 Invertibility [68]
 - 4.3 Translation to sequent calculus [70]
 - 4.4 Unprovability through failed proof search [74]

4.5	Termination of proof search	[76]
	Notes and exercises to Chapter 4	[78]
5	Classical natural deduction	[80]
5.1	Indirect proof	[80]
5.2	Normal derivations and the subformula property	[85]
	Notes and exercises to Chapter 5	[88]
6	Proof search in classical logic	[89]
6.1	Assumptions and cases	[89]
6.2	An invertible classical calculus	[90]
	Notes and exercises to Chapter 6	[95]
7	The semantics of propositional logic	[96]
7.1	Logical truth	[96]
7.2	The semantics of intuitionistic propositional logic	[101]
7.3	Empty tautologies?	[108]
7.4	The completeness of classical propositional logic	[110]
	Notes and exercises to Chapter 7	[112]
PART II LOGICAL REASONING WITH THE QUANTIFIERS		
8	The quantifiers	[115]
8.1	The grammar of predicate logic	[115]
8.2	The meaning of the quantifiers	[120]
	Notes to Chapter 8	[128]
9	Derivations in predicate logic	[129]
9.1	Natural deduction for predicate logic	[129]
9.2	Proof search	[133]
9.3	Classical predicate logic	[143]
	Notes and exercises to Chapter 9	[150]
10	The semantics of predicate logic	[152]
10.1	Interpretations	[152]
10.2	Completeness	[155]
10.3	Interpretation of classical logic in intuitionistic logic	[156]
PART III BEYOND PURE LOGIC		
11	Equality and axiomatic theories	[161]
11.1	Equality relations	[161]
11.2	Sense and denotation	[169]
11.3	Axiomatic theories	[172]
12	Elements of the proof theory of arithmetic	[177]
12.1	The Peano axioms	[177]

- 12.2 Heyting arithmetic [179]
- 12.3 The existence property [185]
- 12.4 A simple-minded consistency proof [188]

PART IV COMPLEMENTARY TOPICS

- 13 Normalization and cut elimination [195]
 - 13.1 Proofs by structural induction [195]
 - 13.2 A proof of normalization [198]
 - 13.3 The Curry–Howard correspondence [205]
 - 13.4 Cuts, their elimination and interpretation [210]
 - 14 Deductive machinery from Aristotle to Heyting [220]
 - 14.1 Aristotle’s deductive logic [220]
 - 14.2 The algebraic tradition of logic [228]
 - 14.3 The logic of Frege, Peano, and Russell [239]
 - 14.4 Axiomatic logic in the 1920s [248]
- Suggestions for the use of this book* [253]
Further reading [256]
Bibliography [258]
Index of names [261]
Index of subjects [262]

Cambridge University Press
978-1-107-03659-8 - Elements of Logical Reasoning
Jan Von Plato
Frontmatter
[More information](#)

Preface

When I was little and Christmas time was approaching, we children knew that there would be two kinds of presents: the soft packages that contained useful but unexciting clothes, and the hard boxes that contained gorgeous new toys. I learned later that the same formula repeats itself often in life, and even in logic. There are the discussions about first principles: what rests on what, what comes first in the end of all analyses, and what it all means – and these are the useful but relatively unexciting soft packages. Then there is the box that is really interesting to open, and that is what I call the deductive machinery of logic – how it all actually works. Others have called it the inferential engine. I believe that logic should not be presented to us just in those soft packages – the hard box has to be there to be opened as well, so that we can find out how logical arguments function. It is a hands-on kind of learning in which one tries and retries things by oneself until the machinery runs smoothly. Then it is the time to discuss the nature of the first principles.

The book begins with a linear form of proofs that I learned from Dag Prawitz' Swedish compendium *ABC i Symbolisk Logik*. Little did I think, back in 1973 when using that text for the first time, that my teaching of elementary logic would one day grow into a comprehensive presentation in the form of a book. Over the years that I have taught logic, students too numerous to be listed here have added to my understanding of how the presentation of the topics should be structured. Next to these experiences, Sara Negri is the person who has contributed decisively to the direction of my work in logic in general. I want to dedicate this book to her and to our proof-theoretical adventure that began quite casually in 1997.

Cambridge University Press sent me, through the good offices of Hilary Gaskin, four anonymous referees' reports with overwhelmingly positive views and comments on the manuscript, which also led to numerous improvements in the presentation.