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PART I

First steps in logical reasoning

1 Starting points

Some of our earliest experiences of the conclusive force of an argument come from school mathematics: Faced with a mathematical proof, however we try to twist the matter, there is no possibility of denying the conclusion once the premisses have been accepted.

Behind the examples from mathematics, there is a more general pattern of 'demonstrative arguments' that is studied in the science of logic. Logical reasoning is applied at all levels, from everyday life to the most advanced sciences. As an example of the former, assume that under some specific conditions, call them A, something, call it B, necessarily follows. Assume further that the conditions A are fulfilled. To deny B under these circumstances would lead to a contradiction, so that either B has to be accepted or at least one of the assumptions revised – or at least that is what the fittest thinker would do to survive.

A remarkable level of complexity is achieved in everyday logical reasoning, even if the principles behind it remain intuitive. We begin our analysis of logical reasoning by the observation that the forms of such reasoning are connected to the forms of linguistic expression used and that these forms have to be made specific and precise in each situation. When this is done, it turns out that a rather limited set of first principles is sufficient for the representation of any logical argument. What appears intuitively as an unlimited horizon of ever more complicated arguments, can be mastered fully by learning these first principles as explained in this book.

1.1 Origins

The idea of logical reasoning appears in the ancient 'science of demonstrative arguments', a terminology from the first logic book ever, Aristotle's *Prior Analytics*. Demonstrative arguments move from what is assumed to be **given** to a **thing sought**. The given can consist of a list of assumptions, the sought of a claim to be proved. A demonstrative science is organized as follows:

1. There are, first, certain **basic concepts** supposed to be understood immediately. Think, as an example, of points and lines in geometry, of a

point being incident with a line, and so on. Next there are **defined concepts**, ones that are not immediately understood. These have to be explained through the basic concepts. A triangle, say, can be defined to be a geometric object that consists of three straight line segments such that any two segments meet at exactly one common endpoint.

2. A second component of a demonstrative science consists of **axioms**. These are assertions the truth of which is immediately evident. We shall soon see some examples of ancient axioms. Next to the axioms, there are assertions the truth of which is not immediately evident. These are the **theorems** and their truth has to be reduced back to the truth of the axioms through a **proof**.

Proofs are things that start from some given **assumptions** and then proceed step by step towards a sought **conclusion**. The most central aspect of such a demonstrative argument is that the conclusion follows necessarily if the assumptions are correct. What the nature of this necessary following is, will be shown by some examples. We shall not, in general, aim at giving any exhaustive coverage of the various concepts that arise, but pass forward through examples. These are situations in which we have a good understanding of things.

1.2 Demonstrative arguments

Let us have a look at some examples of arguments in which the conclusion follows from some given assumptions.

(a) Ancient geometry. There are two types of situations in elementary geometry. In the first, we have some **given objects** of geometry such as points, lines, and triangles, with some prescribed properties. Next there is a **sought object** that has to have a prescribed relation to the given ones. Say, there are two given points, with the property that they are distinct, and the sought object is a triangle with the properties that the line segment with the given points as extremities is the base of the triangle, and that the triangle is equilateral. This is the first result to be established in Euclid's *Elements*, formulated as what is called a construction problem.

In a second kind of situation, there are also given objects with some properties, but the task is to simply prove that these objects have some additional new property. No explicit task of construction is mentioned, but the solution of the task to prove a property often requires intermediate steps of construction of auxiliary geometrical objects.

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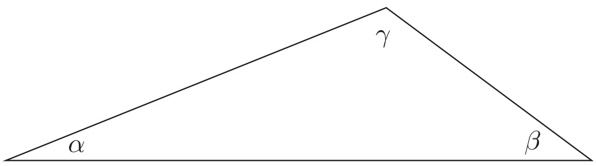
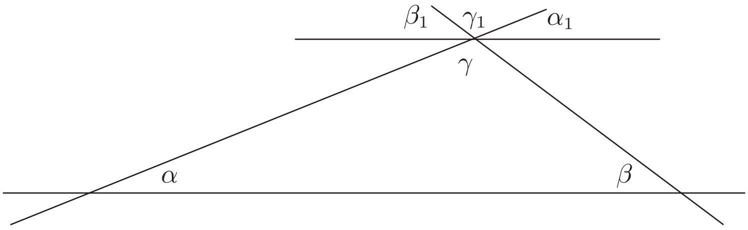
The following example comes from ancient Greek geometry and is of the second kind. Some of the terminology and notation is modern, but the geometrical argument in the example remains the same. Consider any given triangle with the three angles α, β, and γ. Then the sum of these angles is 180°. The result clearly is not anything the truth of which would be immediately evident, but a proof is required. The following figure illustrates the situation:



We have a base of the triangle, limited by the angles α and β. To prove the claim about the sum of the three angles, the sides are next prolonged and a **parallel** to the base drawn through the point that corresponds to angle γ. These are the auxiliary constructions needed:



Symbols have been added to the figure, namely α_1, β_1, and γ_1. We reason as follows: The angle opposite to the original angle γ, namely γ_1, is equal to γ. Next, the line from the original angle α to angle γ intersects the base and the parallel to it. Therefore the angle that is marked by α_1 is equal to the lower left angle α of the triangle. Similarly, β_1 is equal to the lower right angle β of the triangle. We now see that α, β, and γ make up two right angles, or 180°.

The principles that were used in the proof were:

I. *The opposite angles of two intersecting lines are equal.*
II. *If a line intersects two parallel lines, the corresponding angles are equal.*

Both of these were taken to be immediate geometric truths in ancient geometry, i.e., they were considered axioms. If they are accepted, it seems that the claim about the sum of the angles of a triangle would not be a matter of opinion, but a **necessary consequence** of what has been assumed.

In addition to the axioms, what are called **construction postulates** were also used. These include the following, directly from the mentioned standard presentation of Greek geometry, Euclid's *Elements*:

III. *To continue a given finite straight line segment indefinitely.*
IV. *To draw a parallel to a given line through a point outside the line.*

We have a geometric configuration, some properties of which are assumed and with further properties that follow from the construction postulates. We go through in detail the steps that were taken in the proof:

1. By postulate III, the sides of the triangle are continued.
2. By postulate IV, a line parallel to the base is constructed.
3. By axiom II, $\alpha = \alpha_1$.
4. By axiom II, $\beta = \beta_1$.
5. By axiom I, $\gamma = \gamma_1$.
6. $\alpha_1 + \gamma_1 + \beta_1 = 180°$.
7. By 3, 4, and 5, $\alpha_1 + \gamma_1 + \beta_1 = \alpha + \gamma + \beta$.
8. By 6 and 7, $\alpha + \gamma + \beta = 180°$.

Step 8 is based on an axiom that is given in Euclid's *Elements* as:

V. *Any two things equal to a third are equal among themselves.*

Laws of addition have also been used, and in 6 it is seen from the construction that the three angles make up for two right angles.

(**b**) **An example from arithmetic.** One might think that perhaps the objects of geometry are too abstract and our intuitions about their immediately evident properties not absolutely certain. We can take instead the natural numbers: 0, 1, 2, ... Such a number is **prime** if it is greater than one and divisible by only one and itself: 2, 3, 5, 7, 11, 13 .... This series goes on to infinity. **Twin primes** are two consecutive odd numbers that are prime, say 5 and 7, 11 and 13, 17 and 19, and so on. Nobody knows if there is a greatest twin prime pair, or if their series goes on to infinity. Consider three consecutive odd numbers greater than 3. We claim that they cannot each be prime. Assume to the contrary this to be the case, i.e., assume there to be three numbers n, $n + 2$, $n + 4$ such that each is prime and $n > 3$. One out of any three consecutive numbers is divisible by 3 and, thus, one of n, $n + 1$, $n + 2$ is divisible by 3. By our assumption, it can be only $n + 1$. But then also $n + 1 + 3 = n + 4$ is divisible by 3. Our assumption about three primes in succession turned out false.

There is a point in the argument that calls for some attention: It is essential to require that the three odd numbers be greater than 3. We concluded that for any n, one of n, $n+1$, $n+2$ is divisible by 3, and to further conclude that a number divisible by 3 is not prime, it needs to be distinct from 3. Indeed, the sequence 3, 5, 7 is excluded by the requirement.

There does not seem to be any place for opinions about the arithmetical truth established by the above argument. Someone might come and make a clever observation about a proof, especially if it was more complicated than the above example: Maybe something went wrong at some place in the proof. The thing to notice is that the very possibility of having made a mistake presupposes the possibility of the contrary, namely to have a correct proof.

(c) An example from everyday life. At Cap Breton in France, everyone agrees about the following rule: If the wind is hard, it is forbidden to swim. Here comes someone, in agreement with the rule, who also adds: I see some people swimming so I conclude that it is not forbidden to swim, even if I can see that the wind undoubtedly is hard. We could rebuke this someone: You accept that if the wind is hard, it is forbidden to swim. You also accept that the wind is hard. Therefore you accept that it is forbidden to swim, but you also deny it, which makes you contradictory. The person in question might say that it is not forbidden in any legal or moral way to hold contradictory opinions, nor is it a psychological impossibility. Whether it is disadvantageous in the struggle for survival can be debated.

Logical reasoning is based on the acceptance of certain criteria of rationality, such as not to both accept and deny a claim. Such accepting and denying may be hidden: If we accept a claim of a conditional form, say, if something A, then something B, but deny B, acceptance of A will lead to a contradiction as with the Cap Breton bather. The chain of inferences that leads to a contradiction can be so long that we do not necessarily notice anything. However, if a contradiction is pointed out, we should revise some of our assumptions.

1.3 Propositions and assertions

Logical reasoning operates on assumptions and what can be concluded from these. Assumptions and conclusions are things we obtain from **propositions**. We can also call them **sentences**. There is no use in trying to define what sentences are in general. We shall be content to have examples of

complete declarative sentences. Such a sentence expresses a **possible state of affairs**. Again, what possibility or state of affairs etc. is need not be explained in general, but we rest content with good examples. Consider the sentence *It is dark*. Whether this is correct depends on time and place, so let us assume they are fixed. Correctness may also depend on how one defines darkness, astronomically, in civil terms, or what have you, but that is not essential: We have paradigmatic examples of darkness and know what that means and we also know that the notion can be a bit hazy at times. Things such as the natural numbers and their properties would be less hazy, as in: One of 1733, 1735, and 1737 is divisible by 3, something we should believe in by the argument of Section 1.2.

A sentence is something neutral: It merely expresses a possible state of affairs. To make a sentence, call it A, into an assumption, we have to add something to A. This we do by stating: *Let us assume that* A. Similarly, if we conclude A, we actually make a **claim**, namely: A *is the case*. Thus, a sentence is turned into an **assertion** by the addition of an **assertive mood**: *It is the case that* . . . . Other such moods include the interrogative mood for making questions and the imperative mood for giving commands. The sentences that we utter come with a mood that is usually understood by the listener. We do not need to add in front of every sentence *it is the case that* . . . , even if we sometimes do it for emphasis or clarity.

Note also the difference between the **negation** of a sentence, as in *It is not dark*, and the **denial** of a sentence, as in *It is not the case that it is dark*. Denial, like its opposite, namely **affirmation**, is a mood that can be added to a sentence with a negative assertion as a result.

1.4 The connectives

Consider the sentence: *If the wind is hard, it is forbidden to swim*. Its immediate components are two complete declarative sentences *The wind is hard* and *It is forbidden to swim*. These are combined into a **conditional** sentence with the overall structure: **If** . . . , **then** . . . . The word *then* did not occur in the original but is often added in a logical context, to make the structure of conditional sentences clear. Similarly to the conditional, the two components can be combined into the sentences: *The wind is hard* **and** *it is forbidden to swim*, *The wind is hard* **or** *it is forbidden to swim*, *The wind is* **not** *hard*. The combinators in boldface are called **connectives**. We choose a basic stock of connectives and give them names and symbols. For brevity, let the letter A stand for *The wind is hard* and B for *It is forbidden to swim*:

Table 1.1 The propositional connectives

$A \& B$	$A \vee B$	$A \supset B$	$\neg A$

$A \& B$ is the **conjunction** of A and B, to be read A *and* B. $A \vee B$ is the **disjunction** of A and B, to be read A *or* B. $A \supset B$ is the **implication** of A and B, to be read *If* A, *then* B. Finally, $\neg A$ is the **negation** of A, to be read *Not* A.

The sentence A *or* B can be ambiguous: Sometimes A *or* B *or both* is meant, sometimes it is a choice between exclusive alternatives. In propositional logic, disjunction is meant in the inclusive sense, the one that is sometimes written *and/or*.

Further propositional connectives include **equivalence**: A *if and only if* B. The symbolic notation is $A \supset\subset B$. However, the four connectives of Table 1.1 will suffice for us, because other connectives can be defined in terms of them.

The symbolic notation is useful for keeping in mind that the meanings of the logical connectives are fixed and do not depend on the interpretation of a linguistic context by a user of language. The choice of symbols is historical: Most of it comes from Giuseppe Peano in the 1890s, some from Bertrand Russell in the early twentieth century, some later. The implication symbol was originally an inverted letter C, to indicate consequence. When a page was set in a printing office, the letter could be easily inverted and thus the stylized symbol $\supset$ evolved. Conjunction is found on a typewriter keyboard and the capital V disjunction symbol comes from the Latin word *vel* which means and/or. (Latin has also a word for an exclusive disjunction, namely *aut*.) The minus-sign was used for negation.

Logicians after Peano and Russell have made their own choices of symbols. Here is a partial list of symbols that have been used:

Table 1.2 Notational variants of the connectives

Conjunction:	$A \& B$, $A \wedge B$, $A \cdot B$, AB.
Disjunction:	$A \vee B$, $A + B$, $A \mid B$.
Implication:	$A \supset B$, $A \rightarrow B$, $A \Rightarrow B$.
Equivalence:	$A \supset\subset B$, $A \leftrightarrow B$, $A \Leftrightarrow B$, $A \equiv B$.
Negation:	$\neg A$, $-A$, $\sim A$, $\overline{A}$.

When symbolic languages were created they were sometimes accompanied by ideas about a universal language, such as Peano's creation he called

Interlingua. Other similar languages such as Esperanto have been created with the idea of promoting understanding: With a language common to all mankind, wars would end, etc. It is good to remember that part of the motivation for the development of logical languages came from such idealistic endeavours.

The one who contributed most to the development of the basic logical systems, namely Gottlob Frege, called his logical language *Begriffsschrift*, conceptual notation. He added with obvious pride that in it, 'everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated; *nothing is left to guesswork*'.

1.5 Grammatical variation, unique readability

(a) Grammatical variation. The two sentences *If A, then B* and *B if A* seem to express the same thing. Natural language seems to have a host of ways of expressing a conditional sentence that is written $A \supset B$ in the logical notation. Consider the following list:

From A, B follows. A is a sufficient condition for B. A entails B. A implies B. B provided that A. B is a necessary condition for A. A only if B.

The last two require some thought. The **equivalence** of A and B, $A \supset\subset B$ in logical notation, can be read as *A if and only if B*, also *A is a necessary and sufficient condition for B*. Sufficiency of a condition as well as the 'if' direction being clear, the remaining direction is the opposite one. So *A only if B* means $A \supset B$ and so does *B is a necessary condition for A*.

It sounds a bit strange to say that *B is a necessary condition for A* means $A \supset B$. When one thinks of conditions as in $A \supset B$, usually A would be a cause of B in some sense or other, and causes must precede their effects. A necessary condition is instead something that necessarily follows, therefore not a condition in the causal sense.

The conjunction *A and B* in natural language can contain shades of meaning not possessed by the conjunction of propositional logic. In the sentence *John is married and his wife is Mary*, the second conjunct presupposes the first one, as can be seen by considering the sentence with the conjuncts reversed: *John's wife is Mary and he is married.*

Grammatical variation is an aspect of natural language that renders it less monotone, but that is not an issue in logic.

(b) Unique readability. In logic, the symbols are not the essential point, but the uniqueness of meaning of sentences. Let $A, B, C, \ldots$ be sentences. We