How Humans Learn to Think Mathematically

How Humans Learn to Think Mathematically describes the development of mathematical thinking from the young child to the sophisticated adult. Professor David Tall reveals the reasons why mathematical concepts that make sense in one context may become problematic in another. For example, a child's experience of whole number arithmetic successively affects subsequent understanding of fractions, negative numbers, algebra and the introduction of definitions and proof. Tall's explanations for these developments are accessible to a general audience while encouraging specialists to relate their areas of expertise to the full range of mathematical thinking. The book offers a comprehensive framework for understanding mathematical growth, from practical beginnings through theoretical developments, to the continuing evolution of mathematical thinking at the highest level.

David Tall is Emeritus Professor of Mathematical Thinking at the University of Warwick and Visiting Professor at the Mathematics Education Centre, Loughborough University. He is internationally known for his research into long-term mathematical development at all levels, from preschool to the frontiers of research, including in-depth studies explaining mathematical success and failure.

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How Humans Learn to Think Mathematically

Exploring the Three Worlds of Mathematics

DAVID TALL
Emeritus Professor in Mathematical Thinking,
University of Warwick
Visiting Professor, Mathematics Education Centre,
Loughborough University

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To My Family, Friends,

Teachers, Colleagues and Research Students,

Who Made this Book Possible

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Series Foreword

This series for Cambridge University Press is widely known as an international forum for studies of situated learning and cognition. Innovative contributions are being made by anthropology; by cognitive, developmental and cultural psychology; by computer science; by education; and by social theory. These contributions are providing the basis for new ways of understanding the social, historical and contextual nature of learning, thinking and practice that emerges from human activity. The empirical settings of these research inquiries range from the classroom to the workplace, to the high-technology office, and to learning in the streets and in other communities of practice. The situated nature of learning and remembering through activity is a central fact. It may appear obvious that human minds develop in social situations and extend their sphere of activity and communicative competencies. But cognitive theories of knowledge representation and learning alone have not provided sufficient insight into these relationships. This series was born of the conviction that new exciting interdisciplinary syntheses are underway as scholars and practitioners from diverse fields seek to develop theory and empirical investigations adequate for characterizing the complex relations of social and mental life, and for understanding successful learning wherever it occurs. The series invites contributions that advance our understanding of these seminal issues.

Roy Pea
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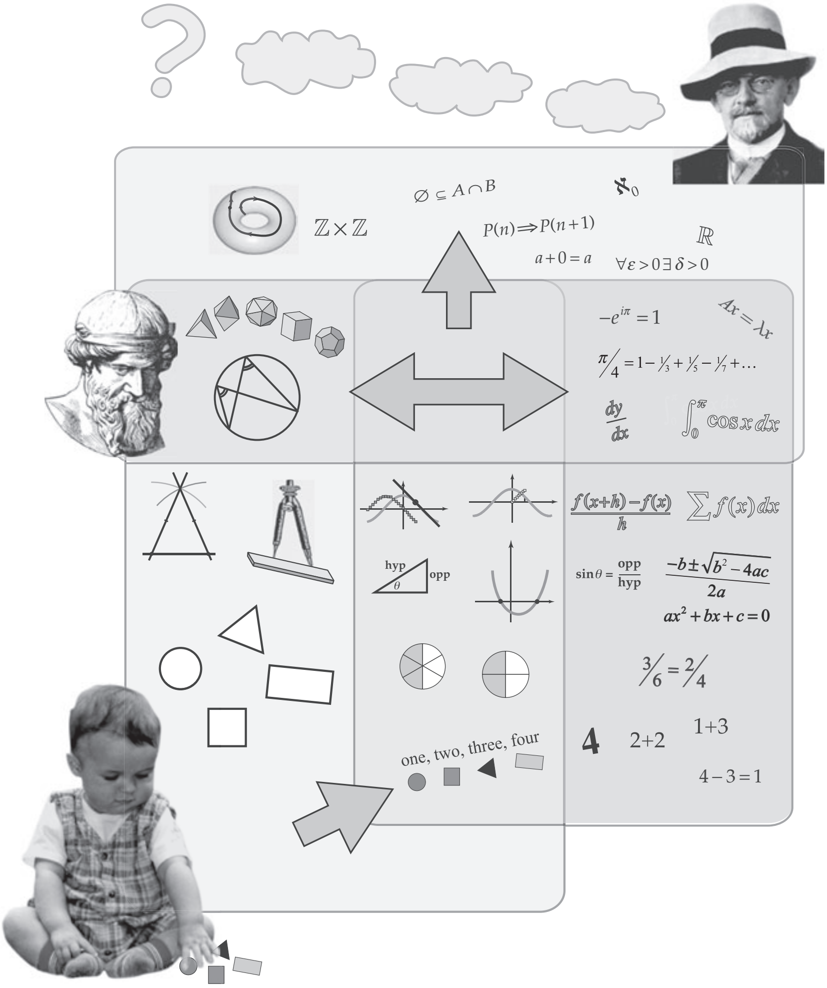

Journeys through three worlds of mathematics.

Preface

I have long been concerned with the struggle to make sense of mathematics and how it is taught and learnt in a world where a few find mathematics an enterprise of great power and beauty, many learn what to do without knowing why and many others find only anxiety and pain.

It has been my good fortune to interact with the teaching and learning of mathematics at every level from preschool to postgraduate research. As I reflected on these experiences and the insights of experts in a range of contexts, it became evident that what happens at each stage of learning is significantly affected by previous experience and has a significant effect on the development of each individual at later stages. This means that it is not sufficient to focus only on a particular level of teaching and learning, as learners at that level will already be affected by what they have met before, and what they learn now will affect future learning. This suggests the need for an overall framework for the development of mathematical thinking so that any individual participating in the enterprise may become aware of a fuller picture. This has the consequence that disputes between different viewpoints may be seen in a new revealing light.

This is, I believe, the first book to focus on the full framework of mathematical thinking as it develops from birth through to adulthood and on to the frontiers of research. It addresses very different theoretical and practical viewpoints and, though I could have decided to focus on a specific readership using technical terms related to a particular community of practice, my experience counsels me that there is a major problem in getting different communities to speak to each other. So I decided to address the book to everyone with a stake in the teaching and learning of mathematics. This includes teachers, mathematicians, educators and curriculum designers, with consequences for parents, politicians and learners, and links to other disciplines in psychology, philosophy, history, cognitive science, constructivism and so on.

This requires taking account of the differing ways in which ideas are formulated in various communities. For instance, if I speak of 'formal' mathematics, a mathematician will think of mathematics designed in terms of set-theoretic definitions and formal proof, whereas an educator following the theory of Piaget may think 'formal' refers to his notion of 'formal operational thinking'. Another reader may think of 'formal' as referring to the formal use of general principles such as 'do the same to both sides' in solving equations.

To attend to these differences I follow the lead of my late supervisor and friend, Richard Skemp, who used two words to formulate special ideas such as 'instrumental understanding' or 'relational understanding'. This juxtaposition of two familiar words in a new way signals to the reader that the terminology is intended to have a special meaning. The word 'understanding' has a general meaning, which may differ subtly for different readers, while the adjective 'instrumental' or 'relational' qualifies the meaning in a more technical way.

My friend and colleague Shlomo Vinner similarly used double-word definitions to describe 'concept definition' and 'concept image' to evoke the general idea of 'concept' in two distinct ways, one based on the definition of the concept given in mathematics, and the other based on the personal image of the concept that an individual has in mind.

I use this simple technique to formulate fundamental ideas to encourage readers from different backgrounds to reflect on the broad theoretical framework and its related empirical evidence. For instance, I need a term to encompass the richness of a meaning of a mathematical concept that becomes more sophisticated as the learner becomes more aware of subtler aspects. I term the full richness of such an idea a 'crystalline concept'. The combination of 'crystalline' and 'concept' is a signal to the reader that this term has a special use. Not only is it a concept, but it also has internal links that hold its various parts together with strong bonds that are a matter of fact within the given context. Whole numbers are crystalline concepts where $2 + 3$ is 5 and cannot be 6. Likewise 5 take away 3 is 2 and cannot be otherwise. Meanwhile, in Euclidean geometry, a triangle with two equal sides must have equal angles and vice versa. In each case the relationships are an essential part of the context.

Crystalline concepts give mathematics its coherent structure. They take on their full meaning as the individual becomes attuned to more sophisticated ideas, but a young child can develop a sense of these relationships at an early stage and the learner can be encouraged to relate ideas in more coherent ways throughout the learning of mathematics.

The maturation of mathematical thinking in different individuals depends on their genetic makeup and on their successive experiences as they learn mathematics, or, more simply, in terms of nature and nurture. By analyzing how children and older students attempt to make sense of successive mathematical concepts, an overall picture of the growth of mathematical thinking emerges. It reveals how humans build ideas through *perception*, *operation* and increasingly subtle *reasoning*, using mathematical symbols and subtle developments in language.

This reveals a deep foundation that is based on what may be termed *the sensori-motor language of mathematics* that underlies three distinct forms of mathematical development, one based on the perception of objects in the world that leads to visual imagery and thought experiments, another based on operations such as counting that lead to number concepts and more sophisticated symbolic developments, and a third based on increasingly sophisticated reasoning that culminates in the formal mathematics of set-theoretic definition and formal proof.

Long-term development depends on making sense of successive levels of sophistication. Mathematics is often considered to be a logical and coherent subject, but the successive developments in mathematical thinking may involve a particular manner of working that is supportive in one context but becomes problematic in another. For example, in everyday life, and in dealing with whole numbers, taking something away always leaves less. But taking away a negative number leaves more, and strange new ideas arise, such as 'two minuses make a plus'.

This phenomenon occurs throughout the long-term development of mathematics as some supportive ways of working in one context continue to work in a subsequent context while other aspects become problematic. Emotion enters into the development as supportive aspects give pleasure and encourage generalization while problematic aspects impede progress. Some who make sense of mathematics at one level and feel confident about the future may enjoy tackling new problems, whereas others, who begin to feel that the mathematics does not make sense, may either take the alternative route to learn how to perform routines without attempting to understand them or, worse still, fall into a downward spiral of anxiety and failure.

An outline of the full theory is presented in Chapter 1. Chapters 2 to 8 are designed to be useful for teachers of mathematics at all levels and cover school mathematics and its consequences. A course for school teachers could usefully concentrate on these chapters, which include an insight into the transition to more formal axiomatic thinking.

After an interlude relating the theory to the historical development of mathematics, Chapters 10 to 14 move on to more advanced topics appropriate for university level, while allowing non-experts to gain a sense of the full range of mathematical thinking.

The final chapter reflects on the overall framework and its relationships with other theoretical frameworks. In particular, by using the observation that supportive aspects in one context may be problematic in another, it reveals new ways of blending different theories together. Rather than indulge in a polemic argument about which of various theories is to be preferred, it reflects on the use of theories being appropriate in different contexts and suggests that a range of conflicting theories have valuable aspects that can be blended together to make more coherent sense. By focusing on foundational ideas, it seeks a framework that applies not only to the personal development of differing individuals from child to adult, but also to the cultural evolution of mathematics in history, and towards the evolution of theories of mathematical thinking in the future.

Acknowledgements

This book is based on thirty years of research and development that depended on the active participation of others. My son, Nic, was my inspiration when, as a boy of five years old, he told me of his ideas on infinity that were far beyond what I expected of such a young child.[1] From then on I was always aware of the very different ways in which children develop.

I had the benefit of two of the greatest thinkers in their subjects as supervisors for my doctorates in mathematics and in the psychology of education: Sir Michael Atiyah, Order of Merit, former President of the Royal Society, awarded the Fields Medal and the Abel Prize for his distinguished research in mathematics, and Professor Richard Skemp, author of *The Psychology of Learning Mathematics* and a major force in the development of the new field of mathematics education. Both feature significantly in the text, as do my colleagues and research students, in particular Eddie Gray, who shared with me the best years of my academic life, as researcher and co-author with his rich insights into the thinking of young children, and as co-supervisor of many of our research students.

Various colleagues have shared developments with me: Professor Richard Skemp, a constant source of inspiration, with a wealth of original ideas on 'instrumental and relational understanding', the emotional affects of goals and anti-goals, and distinct modes of building and testing mathematical concepts; Professor Shlomo Vinner for his ideas on 'concept image'; Dr. Eddie Gray on the flexible idea of 'procept', dually representing process and concept; Professor Efraim Fischbein and Professor Dina Tirosh on concepts of infinity; Dr. Tony Barnard on his theory of 'cognitive units', which became the theory of 'thinkable concepts' and, more significantly, of 'crystalline concepts' in this book; Professor John Pegg, from

[1] Tall (2001).

whom I learnt to link van Hiele theory in geometric development with SOLO taxonomy used for analyzing qualities of learning; and Dr. Anna Poynter, with whom I discovered the three worlds of mathematics through her research on the visualization and symbolism of vectors, and her student, Joshua Payne, who explained to us his idea that 'the sum of two free vectors is the unique free vector that has the same *effect*.'

In many ways, a simple idea can trigger off a completely new train of thought. The notion of 'effect' gave rise to the link between embodied action and compressed symbolism that was the key turning point in the formation of the whole theory of 'three worlds of mathematics' – from embodiment and symbolism in school mathematics to the formal mathematics taught in universities and developed at the frontiers of research. Such ideas stimulate hypotheses that can then be tested in empirical research and built into coherent theories.

Other colleagues who had fundamental ideas that significantly changed my thinking include Rolph Schwarzenberger, Christopher Zeeman, Ian Stewart, David Fowler, Bernard Cornu, Gontran Ervynck, Walter Milner, Nicholas Herscovics, David Wheeler, Joel Hillel, James Kaput, Dick Lesh, Pat Thompson, John Mason, Tommy Dreyfus, Ted Eisenberg, Uri Leron, Keith Schwingendorf, David Feikes, Ed Dubinsky, Anna Sfard, Pessia Tsamir, Guershon Harel, Ivy Kidron, Gary Davis, Adrian Simpson, Janet Duffin, Luiz Carlos Guimarães, Gila Hanna, Bill Byers, Stephen Hegedus, Masami Isoda, Nellie Verhoef, Mikhail Katz, Boris Koichu, Walter Whiteley and many others too numerous to name.

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I have also had the long-term support of my wife Sue and our children, Becki, Chris and Nic and their children, Lawrence, Zac, James, Emily and Simon, who have talked to me about mathematics over a lifetime.

I dedicate this book to all those mentioned above.

Illustration Credits

The illustrations in this book were prepared by the author with the exception of the following:

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