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Prelude

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1 About this Book

Mathematics is a subject with patterns that generate enormous pleasure for some and problems that cause impossible difficulties for others. The situation is made more complicated by different views of what mathematics is and how it should be taught. This book takes a journey from the early conceptions of a newborn child to the frontiers of mathematical research. Its purpose is to present a framework that enables everyone with an interest in mathematical thinking, at any level, to communicate with others in a manner appropriate for their needs. At its foundation is the most fundamental question of all:

> How is it that humans can learn to think mathematically in a way that is far more subtle than the possibilities available for other species?

By focusing on foundational issues and relating them to the long-term development of the subject, it becomes possible to express general ideas at all levels within a single framework, from the ways in which we make sense of the world around us through our perceptions and actions, to the development of more sophisticated ideas using language and symbolism.

Contrary to common belief, new levels of mathematical thinking are not necessarily built consistently on previous experience. Some experiences at one level may be supportive at the next but others may be problematic. For instance, number facts from whole number arithmetic continue to be supportive in fractions and decimals but the experience of multiplying whole numbers sets an expectation that the product is always larger. This becomes problematic when multiplying fractions. Everyday experience tells us that 'taking something away leaves something smaller.' This works for whole numbers and fractions, but it becomes problematic when subtracting a negative number. Over time, supportive aspects encourage progress and give pleasure, while problematic aspects may cause frustration and anxiety that can severely impede learning in new contexts. As differing individuals respond in varying ways to their experiences, there arises a wide spectrum of attitude and progress in making sense of mathematics with long-term consequences.

The foundational ideas in this framework prove to be applicable not only in the teaching and learning of mathematics, but also in the study of its historical development. Even expert mathematicians begin their lives as newborn children and need to develop their mathematical ideas to mature levels in their own cultural and professional environments.

This chapter lays out all the main ideas of the framework, which are then considered in greater detail in the remainder of the book.

1. Children Thinking about Mathematics

John, aged six, sat anxiously at the back of his class as his teacher called out the problems. His page had the numbers one to ten down the left-hand side ready for ten sums in the very first Key Stage One Test in the English National Curriculum. 'Four plus three', called the teacher firmly. Her instructions required students to respond to a question every five seconds. John held out four fingers on his left hand and three on his right and began to count them, pointing at his four left-hand fingers with his right index finger, saying silently, 'one, two, three, four', switching to his right hand, pointing with his left index finger, 'five, six, sev …'. 'Six plus two!' said the teacher. John panicked. He did not have time to write his first sum down and turned his attention to the second. Six plus two is: 'one, two, three, four, five, six, …'. Again his thought was interrupted as the teacher called: 'Four plus two!' John managed this one: the answer was six. He started to write it down, but now he didn't know which number question he was on and wrote it in the space beside the number two. 'Five take away two.' John wrote 'three' in the space beside the number three. So it went on, as he sometimes failed to complete the sum in the given five seconds and sometimes, when he completed the problem in time, he didn't know where to write the answer. He failed his Key Stage One test, feeling glumly that he would never do well in mathematics. It was just too complicated.[1]

In the same school, Peter, not yet five years old, was given a calculator that enabled him to type in a sum such as '4 + 3' on one line and then, when

[1] Eddie Gray and I observed and videoed this episode during a study of how young children perform arithmetic operations in Gray (1993) and Gray & Tall (1994).

he pressed the 'equals' key, the answer was printed on the next line as '7'. He and several friends were asked to use these calculators to type in a sum whose answer was '8'. His friends typed in sums such as 4 + 4 or 7 + 1 or 10 – 2, all of which they could also do practically by counting fingers or objects.

Peter typed in the sum 1000000 – 999992. He knew this was 'a million take away nine hundred and ninety nine thousand, nine hundred and ninety two'. But, of course, he had never counted a million. Just think how long it would take! He could start briskly with 'one, two, three, four, five …' and keep a moderate pace with 'one hundred and eighty seven, one hundred and eighty eight, one hundred and eighty nine, …' but he would be really struggling with 'one hundred and eleven thousand two hundred and seventy eight, one hundred and eleven thousand two hundred and seventy nine, one hundred and eleven thousand two hundred and eighty'.

Peter's ideas arose not directly from counting experiences, but from his knowledge of number relationships. He had clearly been given a great deal of support with number concepts outside of school. Even so, his knowledge was exceptional. He knew about place value: that 10 represented ten, 100 is a hundred, 1,000 a thousand and 1,000,000 a million. He knew about tens of thousands, hundreds of thousands and that a million was a thousand thousands. He knew that 9 and 1 makes 10, 39 and 1 makes 40, 99 and 1 makes a hundred and 999,999 and 1 makes a million. For him it was straightforward to see that just as the sum 92 and 8 gives 100, the sum of 992 and 8 gives a thousand and 999,992 and 8 is a million.[2]

Here we have two children in the same school at about the same age thinking very differently. Can we find a single theoretical framework that encompasses both? Do they both go through the same kind of development, but one happens to achieve more success than the other? How can we formulate a single theory that enables us to improve the teaching and learning of mathematics in a world in which some find mathematics an amazing thing of beauty while others find it a source of problematic anxiety?

To seek a unified theory of the development of mathematical thinking, this book studies the underlying development of mathematical ideas, some of which make sense and support more sophisticated mathematical thinking and some of which are problematic and impede progress.

[2] This episode was recorded by Eddie Gray as part of the same study in the same school.

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2. The Long-Term Development of Mathematical Ideas

Mathematical thinking uses the same mental resources that are available for thinking in general. At its foundation is the stimulation of links between neurons in the brain. As these links are alerted, they change biochemically and, over time, well-used links produce more structured thinking processes and more richly connected knowledge structures.[3] The strengthening of useful links between neurons provides new and more immediate paths of thought, so that processes that occur in time – such as counting to add numbers together to get '3 + 2 is 5' – are shortened to operate without counting, so that '3 + 2' immediately outputs the result '5'. This involves a *compression* of knowledge in which lengthy operations are replaced by immediate conceptual links.

The long-term development of mathematical thinking is consequently more subtle than adding new experiences to a fixed knowledge structure. It is a continual reconstruction of mental connections that evolve to build increasingly sophisticated knowledge structures over time.

Geometry begins with the child playing with objects, recognizing their properties through the senses and describing them using language. Over time, the descriptions are made more precise and used as verbal definitions to specify figures that can be constructed by ruler and compass and eventually the properties of figures can be related in the formal framework of Euclidean geometry. For those who study mathematics at university, this may be further generalized to different forms of geometry, such as non-Euclidean geometries, differential geometry and topology. (Advanced topics featuring here, will also be given elementary explanations for the general reader in later chapters.)

The learning of arithmetic follows a different trajectory, starting not with a focus on the *properties* of physical objects, but on *actions* performed on those objects, including counting, grouping, sharing, ordering, adding, subtracting, multiplying and dividing. These actions become coherent mathematical operations and symbols are introduced that enable the operations to be performed routinely with little conscious effort. More subtly, the symbols themselves may be seen not only as operations to be performed but also compressed into mental number concepts that can be manipulated in the mind.

[3] The term 'knowledge structure' may have various connotations in cognitive science, philosophy and other disciplines. Here I refer broadly to the relationships that exist in a particular context or situation, including various links between concepts, processes, properties, beliefs and so on.

Young children are introduced to counting physical objects to develop the concept of number and to learn to calculate with numbers. As they learn to count, they will find that 7 + 2 calculated by counting 2 after 7 to get 'eight, *nine*' is far easier than 2 + 7 by counting 7 after 2 as 'three, four, five, six, seven, eight, *nine*'. Initially it may not be evident that addition by counting is independent of order, but when this is related to the visual layout of objects placed in various ways, properties of arithmetic emerge, such as addition and multiplication being independent of order of operation, and multiplication being distributive over addition. These observations may be formulated as the 'rules of arithmetic' offering a basis for symbolic proof. At a more advanced level, the whole numbers may be formulated as a list of axioms (the Peano Postulates) from which familiar properties of arithmetic may be proved as formal theorems.

Measurement also develops out of actions: measuring lengths, areas, volumes, weights and so on. These quantities can be calculated practically using fractions or to any desired level of accuracy using decimals. Numbers can be represented as points on a number line and formulated at university level as an axiomatic system (a complete ordered field).

Algebra builds on the generalized operations of arithmetic with symbolic manipulations following the rules observed in arithmetic. Algebraic functions may be visualized as graphs, and later algebraic structures may be formulated in various axiomatic systems (such as groups, rings and fields).

Likewise, concepts in the calculus can be expressed visually and dynamically as the changing slope of a graph and the area under a graph, which may be approximated by numerical calculations or expressed precisely through the symbolic formulae for differentiation and related techniques for integration. At university these ideas may be expressed axiomatically in the formal theory of mathematical analysis.

Vectors are introduced as physical quantities with magnitude and direction, written symbolically as column vectors and matrices, and later reformulated axiomatically as vector spaces.

Probability begins by reflecting on the repetition of physical and mental experiments to think how to predict the likely outcome, then performing specific calculations to calculate the probability numerically, and later formulating the principles axiomatically (as a probability space).

These developments incorporate three distinct forms of knowledge. The first involves the study of objects and their properties, leading to mental imagery described in language that grows increasingly subtle. The second grows out of actions that are symbolized and develop into operations in arithmetic and algebra compressed into mental objects such as numbers and algebraic expressions that can be used to formulate and solve problems using operational symbolism. Both develop initially through practical experiences at home and in school and develop through the use of more theoretical definitions and deductions in school.

The third form of mathematical knowledge flowers in the formal approach to pure mathematics encountered at university.

The full framework grows in sophistication from the activities of the child to the frontiers of mathematical research. (Figure 1.1.)

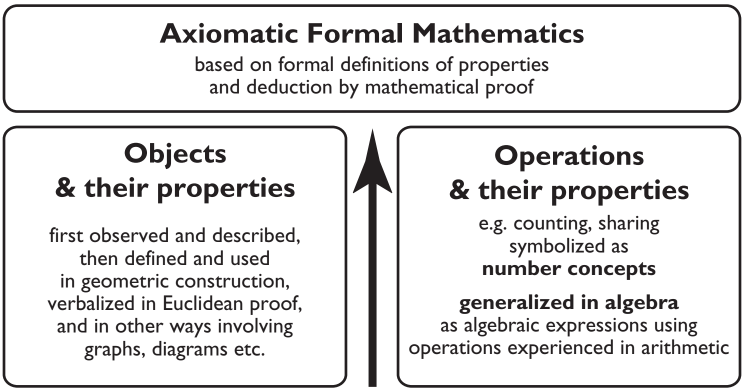

Figure 1.1. An initial outline of three forms of knowledge in mathematics.

3. Existing Theoretical Frameworks

We are already privileged to have many frameworks available to provide an overview of human development in general and mathematics in particular. The father figure of modern developmental psychology, Jean Piaget[4], formulated a stage theory for the long-term development of the child through the pre-language *sensorimotor* stage, a *preoperational* stage in which children develop language and mental imagery from a personal viewpoint, a *concrete-operational* stage wherein they develop stable conceptions of the world shared with others, and a *formal-operational* stage developing the capacity for abstract thought and logical reasoning.

Jerome Bruner[5] classified three modes of human representation and communication: *enactive* (action-based, using gestures), *iconic* (image-based

[4] References on Piaget's Stage Theory are numerous. See, for example, Baron et al. (1995), pp. 326–9.
[5] Bruner (1966), pp. 10, 11.

using pictures and diagrams) and *symbolic* (including language and mathematical symbols).

Efraim Fischbein[6] focused on the development of mathematics and science, and formulated three different approaches, which he called *intuitive*, *algorithmic* and *formal*.

Each of these frameworks presents a long-term development from physical perception and action, through the development of symbolism and language and on to deductive reasoning. They also formulate different ways of building specific concepts. Bruner and Fischbein differ in detail, but both see a broad conceptual development in which the enaction and iconic imagery of Bruner relates to the intuition of Fischbein while Bruner's symbolic mode of operation refers not only to language, but also explicitly to two particular forms of symbolism in arithmetic and logic (relating respectively to mathematical algorithms and formal proof).

Piaget complements his global stage theory by formulating several ways in which new concepts are constructed. The first is *empirical abstraction* through playing with objects to become aware of their properties (for instance, to recognize a triangle as a three-sided figure and to distinguish this from a square or a circle).

The second is *pseudo-empirical abstraction* through focusing on *actions* on objects. This plays a major role in arithmetic where operations such as counting and sharing lead to concepts such as number and fraction.

He also formulates *reflective abstraction* where operations at one level become mental objects of thought at a higher level. This has proved to be fruitful in describing how addition becomes sum, repeated addition becomes product, and, more generally, an operation such as 'double a number and add six' becomes an algebraic expression ($2x + 6$) that is both a process of evaluation and a thinkable algebraic object that can be manipulated to solve problems. Reflective abstraction is essentially a succession of higher-level extensions of pseudo-empirical abstraction.

By analogy, there is a fourth type of abstraction that generalizes empirical abstraction of the properties of physical objects, to imagine mental objects that can exist only in the mind, such as points that have no size and straight lines that have a length but no width. This may be termed *Platonic abstraction* as it forms Platonic mental objects by focusing on the essential properties of figures.[7] (Figure 1.2.)

6 Fischbein (1987).

7 This idea of three (or four) ways of constructing mathematical concepts was proposed in Gray & Tall (2001).

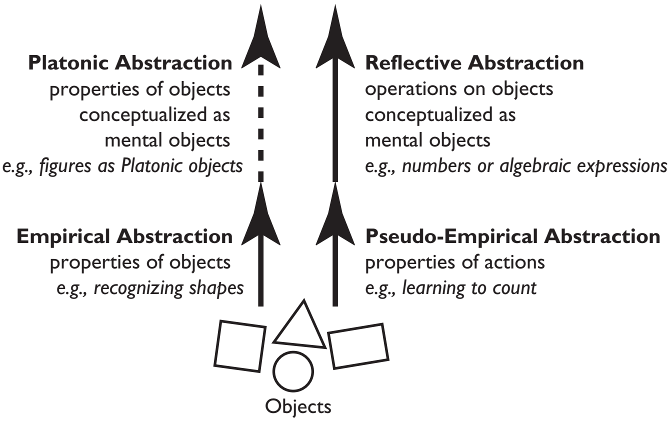

Figure 1.2. Piagetian and Platonic abstraction.

These four types of abstraction belong naturally to two long-term developments, one building from the properties of objects (empirical and Platonic abstraction), and the second from actions on objects (pseudo-empirical and reflective abstraction). These two developments relate directly to the first two forms of long-term development in mathematical thinking formulated earlier. The first focuses on the structure of objects, the second on actions that become operations that are symbolized as mental objects such as numbers and algebraic expressions. I shall refer to these as *structural abstraction* and *operational abstraction*. (Figure 1.3.)

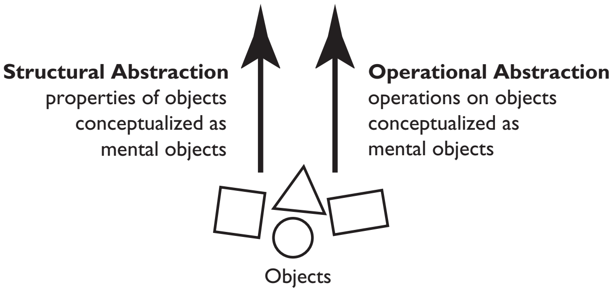

Figure 1.3. Long-term abstraction.

These ideas relate to the vision of Pierre van Hiele's *Structure and Insight*[8] in geometry, and Anna Sfard's formulation of operational and

[8] Van Hiele (1986).