1
Fundamentals

This chapter reviews four important mathematical concepts and techniques that will be helpful in many quantitative problems you're likely to encounter in a college-level introductory astronomy course or textbook. As with all the chapters in the book, you can read the sections within this chapter in any order, or you can skip them entirely if you're already comfortable with this material. But if you're working through one of the later chapters and you find that you're uncertain about some aspect of unit conversion, the ratio method, rate problems, or scientific notation, you can turn back to the relevant section of this chapter.

1.1 Units and unit conversions

One of the most powerful tools you can use in solving problems and in checking your solutions is to consistently include *units* in your calculations. As you may have noticed, among the first things that physics and astronomy professors look for when checking students' work is whether the units of the answer make sense. Students who become adept at problem-solving develop the habit of checking this for themselves.

Understanding units is important not just in science, but in everyday life as well. That's because units are all around you, giving meaning to the numbers that precede them. Telling someone "I have a dozen" is meaningless. A dozen what? Bagels? Minutes to live? Spouses? If you hope to communicate information about quantities to others, numbers alone are insufficient. Nearly every number must have units to define its meaning. So a very good habit to start building mastery is to always include the units of any number you write down.

Of course, some numbers are inherently "unitless." As an example of such a number, consider what happens when you divide the mass of the Sun

(2×10^{30} kg) by the mass of the Earth (6×10^{24} kg) in order to compare their values. The result of this division is approximately 333,333. Not 333,333 kg, just 333,333, because the units of kilograms in the numerator and denominator cancel, as explained later in this section. This unit cancellation happens whenever you divide two numbers with the same units, so you'll see several unitless numbers in Section 1.2 of this chapter.

If keeping track of units is the vital first step in solving astronomy problems, knowing how to reliably convert between different units is a close second. When you travel to a country that uses a different currency, you learn firsthand the importance of unit conversions. If you come upon a restaurant offering a full dinner for 500 rupees, is that a good deal? You'll have to do a unit conversion to find out. And to do that conversion, you'll need two things: (1) a conversion factor between currencies, such as those shown in Figure 1.1; and (2) knowledge of how to use conversion factors.

To understand the process of unit conversion, it's best to start with simple cases using everyday units, because you probably have an intuitive sense of how to perform such conversions. For example, if a movie lasts 2 hours, you know that is 120 minutes, because there are 60 minutes in 1 hour. But think about the process you used to convert hours to minutes: you intuitively multiplied 2 hours by 60 minutes in each hour.

Unfortunately, unit conversion becomes less intuitive when you're using units that are less familiar to you, or when you're using large numbers that can't be multiplied in your head. In such cases, students sometimes resort to guessing whether to multiply or divide the original quantity by the conversion factor. After a short discussion of conversion factors, we'll show you a foolproof method for setting up any unit conversion problem that will ensure you always know whether to multiply or divide.

1.1.1 Conversion factors

So exactly what *is* a conversion factor? It's just a statement of the equivalence between expressions with different units, and that statement lets you translate between those units in either direction. How can two expressions with different numbers be equivalent? Well, the distance represented by 1 meter is exactly the same as the distance represented by 100 cm. So it's the *underlying quantity* that's the same, and that quantity is represented by the *combination* of the number and the unit.

This means that a conversion factor is always a statement that some number of one unit is equivalent to a different number of another unit. Conversion factors are usually written in one of two ways: either as an equivalence relation

Figure 1.1 Currency exchange rates on a bank board. Each entry is a conversion factor between one unit and another.

or as a fraction. For example, 12 inches of length is equivalent to 1 foot, 60 minutes of time is equivalent to 1 hour, and the astronomical distance unit of 1 parsec (pc) is equivalent to 3.26 light years (ly). Each of these conversion factors can be expressed in an equivalence relation, which we signify using a double-headed arrow ($\leftrightarrow$):

$$12 \text{ in} \leftrightarrow 1 \text{ ft}, \qquad 1 \text{ hr} \leftrightarrow 60 \text{ min}, \qquad 3.26 \text{ ly} \leftrightarrow 1 \text{ pc}.$$

For convenience, one of the numbers in a conversion factor is often chosen to be 1, but it doesn't have to be. For example, 36 inches $\leftrightarrow$ 3 feet is a perfectly valid conversion factor.

It is convenient to represent the conversion factor as a fraction, with one set of units and its corresponding number in the numerator, and the other set in the denominator. Representing the example conversion factors shown above as fractions, you have

$$\frac{12 \text{ in}}{1 \text{ ft}} \text{ or } \frac{1 \text{ ft}}{12 \text{ in}}, \qquad \frac{60 \text{ min}}{1 \text{ hr}} \text{ or } \frac{1 \text{ hr}}{60 \text{ min}}, \qquad \frac{3.26 \text{ ly}}{1 \text{ pc}} \text{ or } \frac{1 \text{ pc}}{3.26 \text{ ly}}.$$

Because the two quantities in the conversion factor must represent the same amount, representing them as a fraction creates a numerator and a denominator that are equivalent, and thus the intrinsic *value* of the fraction is 1. You can multiply other values by this fraction with impunity, since multiplying any quantity by 1 does not change the amount – but it does change the way it looks. This is the goal of unit conversion: to change the units in which a quantity is expressed while retaining the underlying physical quantity.

Exercise 1.1. Write the following equivalence relations as fractional conversion factors:
1 in $\leftrightarrow$ 2.54 cm, 1.6 km $\leftrightarrow$ 1 mile, 60 arcmin $\leftrightarrow$ 3,600 arcsec.

1.1.2 Setting up a conversion problem

The previous section explains *why* unit conversion works; here's a foolproof way to do it:

- Find the conversion factor that contains both units – the units you're given and the units to which you wish to convert.
- Write the expression you're given in the original units followed by a $\times$ symbol followed by the relevant conversion factor in fractional form.
- Multiply all the numbers and all the units of the original expression by the numbers and the units of the conversion factor. Grouping numbers and terms allows you to treat them separately, making this step easier.

You can see this method in action in the following example.

Example: Convert 1,000 minutes to hours.

The fractional forms of the relevant conversion factor (that is, the conversion factor containing hours and minutes) are $\frac{1\text{ hr}}{60\text{ min}}$ and $\frac{60\text{ min}}{1\text{ hr}}$. But how do you know which of these to use? Both are proper conversion factors, but one will help you solve this problem more directly.

To select the correct form of the conversion factor, look at the original units you're given. If those units are standing alone (as are the units of minutes in the expression "1,000 minutes"), use the conversion factor with the units you're trying to get rid of in the denominator and the units that you're trying to obtain in the numerator. That way, when you multiply, the units you don't want will cancel, and the units you want will remain. This works because you can cancel units that appear in both the numerator and the denominator of a fraction in the same way you can cancel numerical factors.

In this example, since the units you're given (minutes) appear standing alone and you want to convert to units of hours, the correct form of the conversion factor has minutes in the denominator and hours in the numerator. That factor is $\frac{1\text{ hr}}{60\text{ min}}$. With that conversion factor in hand, you're ready to write down the given quantity in the original units and multiply by the conversion factor. Here's how that looks with the $\boxed{\text{conversion factor}}$ boxed:

$$1{,}000\text{ min} \times \boxed{\frac{1\text{ hr}}{60\text{ min}}}.$$

To simplify this expression, it helps to realize that there is an implicit multiplication between each number and its unit, and to remember that multiplication is commutative – so you can rearrange the order of the terms in both the numerator and denominator. That lets you multiply the numerical parts together and the units together, canceling units that appear on both top and bottom. Then you can simplify the numbers and express your answer in whatever units remain uncanceled:

$$1{,}000\text{ min} \times \boxed{\frac{1\text{ hr}}{60\text{ min}}} = \frac{(1{,}000 \times 1)(\cancel{\text{min}} \times \text{hr})}{60\ \cancel{\text{min}}} = \frac{1{,}000\text{ hr}}{60} = 16.7\text{ hr}$$

So a time value of 1,000 minutes represents the same amount of time as 16.7 hours.

Here's another example that uses the common astronomical distance units of parsecs and light years:

Example: Convert 1.29 parsecs, the distance of the closest star beyond our Sun, to light years.

In most astronomy texts, you'll find the conversion factor between parsecs and light years given as 3.26 ly $\leftrightarrow$ 1 pc, or equivalently 0.3067 pc $\leftrightarrow$ 1 ly.

In this case, since the quantity you're given has units of parsecs standing alone, you'll need the fractional conversion factor with parsecs in the denominator and light years in the numerator. Using that factor, your multiplication should look like this, again with the conversion factor boxed:

$$1.29\text{ pc} = 1.29\text{ pc} \times \boxed{\frac{3.26\text{ ly}}{1\text{ pc}}} = \frac{(1.29 \times 3.26)(\cancel{\text{pc}} \times \text{ ly})}{1\ \cancel{\text{pc}}} = \frac{4.21\text{ ly}}{1} = 4.21\text{ ly}.$$

Notice that the original quantity of 1.29 pc may be written as the fraction $\frac{1.29\text{ pc}}{1}$ in order to remind you to multiply quantities in both the numerator and in the denominator. The result of this unit conversion tells you that 4.21 light years represent the same amount of distance as 1.29 parsecs. Thus, the light from the nearest star beyond the Sun (a star called Proxima Centauri) takes over 4 years to travel to Earth.

An additional benefit of this method of unit conversion is that it helps you catch mistakes. Consider what would happen if you mistakenly used the conversion factor upside-down; the units of your answer wouldn't make sense. Here are incorrect setups for the previous two examples:

$$\begin{aligned} 1{,}000 \text{ min} \times \frac{60 \text{ min}}{1 \text{ hr}} &= \frac{(1{,}000 \times 60)(\text{min} \times \text{min})}{1 \text{ hr}} \\ &= 60{,}000 \frac{\text{min}^2}{\text{hr}} (\text{INCORRECT}) \end{aligned}$$

and

$$1.29 \text{ pc} \times \frac{1 \text{ pc}}{3.26 \text{ ly}} = \frac{(1.29 \times 1)(\text{pc} \times \text{pc})}{3.26 \text{ ly}} = 0.40 \frac{\text{pc}^2}{\text{ly}} \text{ (INCORRECT).}$$

Since these units are not the units to which you're trying to convert, you know you must have used conversion factors incorrectly.

Exercise 1.2. Perform the following unit conversions (you can find the relevant conversion factors in most astronomy texts or on the Internet).

(a) Express 12 inches in centimeters.
(b) Express 100 cm in inches.
(c) Express 380,000 km in miles (this is roughly the distance from the Earth to the Moon).
(d) Express 93,000,000 miles in kilometers (this is roughly the distance from the Earth to the Sun).
(e) Express 0.5 degrees in arcseconds (this is roughly the angular size of the full Moon viewed from Earth).

1.1.3 Checking your answer

Whenever you do a unit conversion (or other problems in astronomy, or any other subject for that matter), you should always give your answer a sanity check. That is, you should ask yourself "Does my answer make sense? Is it reasonable?" For example, in the incorrect version of the conversion from minutes to hours, you can definitely tell from the numerical part of your answer that something went wrong. After all, since 60 minutes are equivalent to 1 hour, then for any amount of time the number of minutes must be greater than the equivalent number of hours. So if you were to convert 1,000 minutes to hours and obtain an answer of 60,000 hours, the number of minutes would be smaller than the number of hours. That means these two quantities can't possibly be equivalent, which alerts you to a mistake somewhere.

Of course, if the units are outside your common experience (such as parsecs and light years in the previous example), you might not have a sense of what is or isn't reasonable. But you'll develop that sense with practice, so be sure to always take a step back from your answer to see if it makes sense. And remember that whenever you're converting to a *larger* unit (such as minutes to hours), the numerical part of the answer should get *smaller* (so that the combination of the number and the units represents the same quantity).

Exercise 1.3. How do you know that your answers to each of the unit conversion problems in the previous exercise make sense? Give a brief explanation for each.

1.1.4 Multi-step conversions

Up to this point, we've been working with quantities that have single units, such as meters, hours, or light years. But many problems in astronomy involve quantities with multiple units, such as meters per second or watts per square meter. Happily, the conversion-factor approach works just as well for multi-unit quantities.

Example: Convert from kilometers per hour to meters per second.

Since this problem statement doesn't tell you how many km/hr, you can use 1 km/hr. To convert quantities which involve two units (kilometers and hours in this case), you can use two conversion factors in immediate succession: one to convert kilometers to meters and another to convert hours to seconds. Here's how that looks:

$$\frac{1\ \text{km}}{\text{hr}} \times \boxed{\frac{1000\ \text{m}}{1\ \text{km}}} \times \boxed{\frac{1\ \text{hr}}{3600\ \text{s}}} = \frac{(1 \times 1{,}000 \times 1)(\cancel{\text{km}} \times \text{m} \times \cancel{\text{hr}})}{(1 \times 3{,}600)(\cancel{\text{hr}} \times \cancel{\text{km}} \times \text{s})},$$

$$1\frac{\text{km}}{\text{hr}} = \frac{1{,}000}{3{,}600}\frac{\text{m}}{\text{s}} = 0.28\ \text{m/s}.$$

Alternatively, you could have done two separate conversions in succession, such as km/hr to km/s, and then km/s to m/s.

You may also encounter problems in which you need to break a single conversion into multiple steps. This may occur, for example, if you don't know the conversion factor directly from the given units to the desired units, but you do know the conversions for intermediate units. This is illustrated in the following example:

Example: How many seconds old were you on your first birthday?

Even if you don't know how many seconds are in a year, you can break this problem up into years to days, then days to hours, hours to minutes, and finally minutes to seconds. So to convert between years and seconds, you could do the following:

$$1\,\cancel{\text{yr}} \times \boxed{\frac{365\,\cancel{\text{d}}}{1\,\cancel{\text{yr}}}} \times \boxed{\frac{24\,\cancel{\text{hr}}}{1\,\cancel{\text{d}}}} \times \boxed{\frac{60\,\cancel{\text{min}}}{1\,\cancel{\text{hr}}}} \times \boxed{\frac{60\text{ s}}{1\,\cancel{\text{min}}}} = \frac{(365 \times 24 \times 60 \times 60)\text{ s}}{1}$$

$$= 31{,}536{,}000\text{ s}.$$

By determining that there are about 31.5 million seconds in a year, you've derived the conversion factor between seconds and years. With the fractional conversion factor $\frac{31{,}536{,}000\text{ s}}{1\text{ yr}}$ in hand you can, for example, find the number of seconds in 30 years in a single step:

$$30\,\cancel{\text{yr}} \times \boxed{\frac{31{,}536{,}000\text{ s}}{1\,\cancel{\text{yr}}}} = \frac{30 \times 31{,}536{,}000\text{ s}}{1} = 946{,}080{,}000\text{ s},$$

which is just under 1 billion. This gives you a sense of how large a billion is – you've lived a million seconds when you're about 11.5 days old, but even 30 years later you still haven't lived for a billion seconds.

Exercise 1.4. Perform the following unit conversions.

(a) Convert 60 mph (miles per hour) to meters per second.

(b) Convert 1 day to seconds.

(c) Convert dollars per kilogram to cents per gram (100 cents $\leftrightarrow$ 1 dollar).

(d) Convert 1 mile to steps, assuming 1 step $\leftrightarrow$ 30 inches (there are 1,760 yards in 1 mile, 3 ft in 1 yard, and 12 inches in 1 ft).

1.1.5 Converting units with exponents

Sometimes when doing a unit conversion problem, you will need to convert a unit that is raised to a power. In these cases, you must be sure to raise the conversion factor to the same power, and apply that power to all numbers *and* units in the conversion factor.

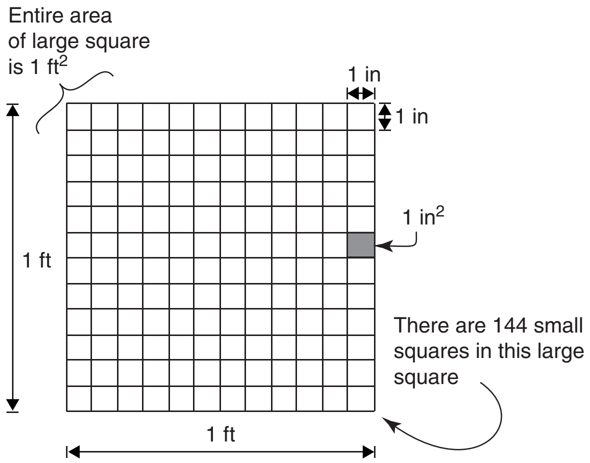

Figure 1.2 One square foot (ft^2), composed of 12 in $\times$ 12 in = 144 in^2.

Example: Convert 1 square foot (1 ft^2) to square inches (in^2).

You already know that there are 12 inches in 1 foot. Feet and inches are both units of one-dimensional length, or *linear* dimension. *Square* feet and inches, however, are units of two-dimensional *area*. The illustration in Figure 1.2 makes it clear that one square foot is not equal to just 12 square inches, but rather 12^2, or 144 square inches.

To perform this unit conversion mathematically, without having to draw such a picture, you'd write:

$$1\text{ ft}^2 = 1\text{ ft}^2\left(\frac{12\text{ in}}{1\text{ ft}}\right)^2 = 1\text{ ft}^2\left(\frac{12^2\text{ in}^2}{1^2\text{ ft}^2}\right) = 1\,\cancel{\text{ft}^2}\left(\frac{144\text{ in}^2}{1\,\cancel{\text{ft}^2}}\right) = 144\text{ in}^2.$$

Notice that when you raise the conversion factor ($\frac{12\text{ in}}{1\text{ ft}}$) to the second power, both the numerical parts and the units, in both numerator and denominator, get squared.

Example: How many cubic centimeters (cm^3) are in 1 cubic meter (m^3)?

You know that there are 100 cm in 1 m, and both centimeters and meters are units of one-dimensional length. A cubic length unit, however, is a unit of three-dimensional volume. When you multiply by the appropriate conversion factor that converts between centimeters and meters, you must raise that factor to the third power, applying that power to all numbers and units separately.

$$1\text{ m}^3 = 1\text{ m}^3\left(\frac{100\text{ cm}}{1\text{ m}}\right)^3 = 1\,\cancel{\text{m}^3}\left(\frac{100^3\text{ cm}^3}{1^3\,\cancel{\text{m}^3}}\right) = 1{,}000{,}000\text{ cm}^3,$$

so there are 1 million cubic centimeters in 1 cubic meter.

Example: Convert 9.8 m/s^2 to km/hr^2.

One conversion factor is needed to convert length from meters to kilometers, and another to convert time from seconds to hours. The time conversion factor needs to be squared, but the length conversion factor does not.

$$9.8\frac{\text{m}}{\text{s}^2} = 9.8\frac{\cancel{\text{m}}}{\cancel{\text{s}^2}}\left(\frac{1\text{ km}}{1{,}000\ \cancel{\text{m}}}\right)\left(\frac{3{,}600\ \cancel{\text{s}}}{1\text{ hr}}\right)^2 = \frac{9.8\times 3{,}600^2\text{ km}}{1{,}000\text{ hr}^2} = 127{,}000\frac{\text{km}}{\text{hr}^2}.$$

Exercise 1.5. Perform the following unit conversions.

(a) How many square feet are in 1 square inch?
(b) Convert 1 cubic foot to cubic inches.
(c) How many square centimeters are in a square meter?
(d) Convert 1 cubic yard to cubic feet (3 feet $\leftrightarrow$ 1 yard).

1.1.6 Compound units

A handful of units that you're likely to encounter in an astronomy class are actually compound units, meaning that they are combinations of more basic[1] units. For example, the force unit of newtons (N) is defined as a mass in kilograms times a distance in meters divided by the square of the time in seconds: 1 N = 1 kg·m/s^2. This means that wherever you see units of newtons (N), you are free to replace that unit with its equivalent, kg·m/s^2, without changing the number in front of the unit. Put another way, you can use 1 N $\leftrightarrow$ 1 kg·m/s^2 to make the conversion factor $\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}$ or its inverse, which you can multiply by your original quantity in order to get it into a new set of units.

The energy unit joules is another example. Energy has dimensions of force (SI units of newtons) times distance (SI units of meters), so 1 J $\leftrightarrow$ 1 N·m.

As one final example of compound units, the power units of watts (W) are defined as energy (SI units of joules) per time (SI units of seconds). Therefore 1 W $\leftrightarrow$ 1 J/s.

Example: Express the compound unit watts in terms of the base units kilograms (kg), meters (m), and seconds (s).

The definition of watts is given just above: energy per unit time, with SI units of joules per second:

$$1\text{ W} \quad \leftrightarrow \quad 1\text{ J/s}.$$

[1] The *base units* you will encounter in this book are those of the International System of Units ("SI"): meters for length, kilograms for mass, seconds for time, and kelvins for temperature. Many astronomers (and some astronomy texts) use the "cgs" system in which the standard units are centimeters for length, grams for mass, and seconds for time.