Dynamic Models for Volatility and Heavy Tails

The volatility of financial returns changes over time and, for the last thirty years, Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models have provided the principal means of analyzing, modelling and monitoring such changes. Taking into account that financial returns typically exhibit heavy tails – that is, extreme values can occur from time to time – Andrew C. Harvey’s new book shows how a small but radical change in the way GARCH models are formulated leads to a resolution of many of the theoretical problems inherent in the statistical theory. The approach can also be applied to other aspects of volatility, such as those arising from data on the range of returns and the time between trades. Furthermore, the more general class of Dynamic Conditional Score models extends to robust modelling of outliers in the levels of time series and to the treatment of time-varying relationships. As such, there are applications not only to financial data but also to macroeconomic time series and to time series in other disciplines. The statistical theory draws on basic principles of maximum likelihood estimation and, by doing so, leads to an elegant and unified treatment of nonlinear time-series modelling. The practical value of the proposed models is illustrated by fitting them to real data sets.

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With Applications to Financial and Economic Time Series

Andrew C. Harvey
University of Cambridge
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Preface

This book sets out a class of nonlinear time series models designed to extract a dynamic signal from noisy observations. The signal may be the level of a series or it may be a measure of scale. Changing scale is of considerable importance in financial time series where volatility clustering is an established stylized fact. Generalized autoregressive conditional heteroscedasticity (GARCH) models are widely used to extract the current variance of a series. However, using variance (or rather, standard deviation) as a measure of scale may not be appropriate for non-Gaussian (conditional) distributions. This is of some importance, because another established feature of financial returns is that they are characterized by heavy tails.

The dynamic equations in GARCH models are filters. Just as the filters for linear Gaussian location models are linear combinations of past observations, so GARCH filters, because of their Gaussian origins, are usually linear combinations of past squared observations. The models described here replace the observations or their squares by the score of the conditional distribution. Furthermore, when modelling scale, an exponential link function is employed, as in exponential GARCH (EGARCH), thereby ensuring that the filtered scale remains positive. The unifying feature of the models in the proposed class is that the asymptotic distribution of the maximum likelihood estimators is established by a single theorem that delivers an explicit analytic expression for the asymptotic covariance matrix of the estimators. Furthermore, the conditions under which the asymptotics go through are relatively straightforward to verify. There is no such general theory for GARCH models: analytic expressions for the asymptotic covariance matrix of the maximum likelihood estimators cannot be found even in the most basic cases, and for some models, most notably EGARCH, there is no asymptotic theory except for very special cases that are never used in practice.

Other properties of the proposed models may be found. These include analytic expressions for moments, autocorrelation functions and multistep forecasts. The properties, particularly for the volatility models, which employ an exponential link function, are more general than is usually the case. For example, expressions for unconditional moments, autocorrelations and the...
conditional moments of multistep predictive distributions can be obtained for absolute values of the observations raised to any power.

The generality of the approach is further illustrated by consideration of dynamic models for non-negative variables. Such models have been used for modelling duration, range and realized volatility in finance. Again, the use of an exponential link function combined with a dynamic equation driven by the conditional score gives a range of analytic results similar to those obtained with the new class of EGARCH models.

Estimating a dynamic level embedded in noise is explicitly an exercise in signal extraction. A general treatment of Gaussian models is based on the state space form and the Kalman filter. When the noise comes from a heavy-tailed distribution, such as Student’s $t$, the filter proposed here can be regarded as an approximation to a filter for the signal plus noise model that can only be obtained by computer simulation techniques, as in Durbin and Koopman (2012). However, its properties are obtained by treating it as a model in its own right. Such a model is said to be observation-driven, as opposed to the unobserved components model, which is parameter-driven. Turning to scale, GARCH models are not usually seen as vehicles for signal extraction, but this is precisely what they are. That this is the case becomes clearer if they are viewed as observation-driven approximations to parameter-driven stochastic volatility models. Indeed, this was part of the original motivation for the formulation of EGARCH models. The development of the class of observation-driven models in this book acknowledges the link with parameter-driven models, and in doing so, it takes a step towards a unified theory of nonlinear time series models.

The book assumes that the reader is familiar with the basic ideas and technicalities of time series. The mathematics is not too demanding given a good understanding of statistical concepts such as conditional distributions and maximum likelihood estimation. Hence it should be accessible to graduate students in the more technical areas of economics and finance, as well as to statisticians. Sections marked with an asterisk (*) are more technical and/or tangential to the main argument and can be skipped without loss of continuity.

The idea of using the score to drive the dynamics in non-Gaussian models is not new, but up to now has had no firm theoretical foundation. The research for this book began in 2008 with a working paper I wrote with a student, Tirthankar Chakravarty, on EGARCH models. At the same time, Siem-Jan Koopman and his co-workers were independently developing a range of score-driven models. They also produced a working paper in 2008. Because Siem-Jan and I have co-authored many papers on unobserved component models, it is perhaps not too surprising that we hit on the same idea, albeit by different routes. One of the difficulties we faced was that the models lacked a convincing asymptotic

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1 Rather than the term dynamic conditional score (DCS) models, which I use here, Creal, Koopman and Lucas (2011) prefer the name generalized autoregressive score (GAS). However, despite the attraction of the acronym, the term ‘autoregressive’ seems to me to convey a more limited dynamic structure than is actually the case.
theory for maximum likelihood estimation. Fortunately, a six-month visit to Carlos III University in Madrid in 2010 provided me with the inspiration to develop the necessary theory. I’m grateful to the Bank of Santander for its support under the Carlos III program for Chairs of Excellence. Further work was done when I was a visiting Fernand Braudel Fellow at the European University Institute in Florence towards the end of 2011. It was there, in the garden of the Villa San Paolo, that Anders Rahbek gave me a memorable tutorial on the finer points of advanced asymptotic theory for time series. I’m grateful to Anders and to all the other colleagues who have provided comments and support during the work on the project. These include Philipp Andres, Tirthankar Chakravarty, Frank Diebold, Rob Engle, Gloria Gonzalez-Rivera, Peter Hansen, Stan Hurn, Ryoko Ito, Siem-Jan Koopman, Alessandra Luati, Mark Salmon, Steve Satchell, Richard Smith, Genaro Sucarrat, Abderrahim Taamouti, Stephen Thielemann and Paolo Zaffaroni. Universities at which the ideas were presented include Oxford, Warwick, Queensland, Monash, Hanover, EUI, Carlos III, Alicante, New York, Columbia and Pennsylvania. Special thanks go to Esther Ruiz at Carlos III and Mardi Dungey at the University of Tasmania, where I spent three weeks in December 2010. Finally I’d like to thank Rosa Matzkin and two anonymous readers for their helpful and constructive comments, and Peihang Lu for editorial assistance.
Acronyms and Abbreviations

ACD  autoregressive conditional duration
ACF  autocorrelation function
AIC  Akaike information criterion
APARCH asymmetric power ARCH
ARCH autoregressive conditional heteroscedasticity
ARIMA autoregressive integrated moving average
BIC  Bayesian information criterion
CAViaR conditional autoregressive value at risk by regression quantiles
CDF  cumulative distribution function
CPI  consumer price index
CV   coefficient of variation
DCC  dynamic conditional correlation
DCS  dynamic conditional score
EGARCH exponential GARCH
ES   expected shortfall
EWMA exponentially weighted moving average
GARCH generalised autoregressive conditional heteroscedasticity
GED  general error distribution
GG   generalised gamma
IF   innovations form
IGARCH integrated GARCH
IID  independent and identically distributed
IRW  integrated random walk
KF   Kalman filter
LIE  law of iterated expectations
LM   Lagrange multiplier
LR   likelihood ratio
MA   moving average
MCMC Markov chain Monte Carlo
MD   martingale difference
MEM  multiplicative error models
MGF  moment generating function
<table>
<thead>
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<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>ML</td>
<td>maximum likelihood</td>
</tr>
<tr>
<td>MMSE</td>
<td>minimum mean square error (estimate)</td>
</tr>
<tr>
<td>MSE</td>
<td>mean square error</td>
</tr>
<tr>
<td>NID</td>
<td>normally and independently distributed</td>
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<tr>
<td>PDF</td>
<td>probability distribution function</td>
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<tr>
<td>PIT</td>
<td>probability-integral transform</td>
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<tr>
<td>QARMA</td>
<td>quasi-ARMA</td>
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<tr>
<td>QML</td>
<td>quasi-maximum likelihood</td>
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<tr>
<td>QQ</td>
<td>quantile-quantile</td>
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<tr>
<td>RMSE</td>
<td>root mean square error</td>
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<td>SD</td>
<td>standard deviation</td>
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<tr>
<td>SE</td>
<td>standard error</td>
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<tr>
<td>SRE</td>
<td>stochastic recurrence equation</td>
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<td>state space form</td>
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<td>structural time series model</td>
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