## Contents

**Preface**

1 Introduction

1.1 What is signal and image analysis? 1
1.2 Why geometric methods? 1
1.3 Applications 4

1.3.1 Image edge detection 4
1.3.2 Image segmentation 6
1.3.3 Diffusion tensor imaging 6
1.3.4 Surface denoising 7
1.3.5 Surface compression 8
1.3.6 Shape skeletonization 9
1.3.7 Shape recognition 10
1.3.8 Networked sensors and data 13

2 Fundamentals of group theory

2.1 Elements of group theory 15

2.1.1 Groups: definitions and examples 15
2.1.2 Homomorphisms of groups 22
2.1.3 Cyclic groups 26
2.1.4 Permutation groups 26
2.1.5 Matrix groups 29

2.2 Topological and symmetry groups 29

2.2.1 Topological spaces 30
2.2.2 Topological groups 32
2.2.3 Isometry between metric spaces 33
2.2.4 Symmetry groups 35

2.3 Geometric groups 36

2.3.1 Introduction to graph theory 36
2.3.2 Geometric groups and Cayley graphs 42

2.4 Symmetry discovery of nonrigid 3D shapes 43

2.4.1 Skeleton path acquisition 44
2.4.2 Endpoints matching 48
## Contents

2.4.3 Symmetry discovery 49  
2.4.4 Symmetric components discovery 50

3 Vector spaces 53

3.1 Vector space theory 53  
3.1.1 Vector spaces over a field 54  
3.1.2 Cartesian product of spaces 59  
3.1.3 Subspaces of vector spaces 59  
3.1.4 Linear independence and bases 60  
3.1.5 Direct sum 62  
3.1.6 Quotient spaces 62

3.2 Linear operators 63  
3.2.1 Isomorphism 68  
3.2.2 Kernel and image 69  
3.2.3 Matrix of a linear operator 71  
3.2.4 Eigenvalues and eigenvectors of linear operators 72  
3.2.5 Eigendecomposition of matrices 73  
3.2.6 Linear functionals and dual space 76

3.3 Inner product spaces 77  
3.3.1 Dot and cross products 77  
3.3.2 Inner product 78  
3.3.3 Orthogonal bases 83  
3.3.4 Orthogonal complements 84  
3.3.5 Orthonormal bases 85  
3.3.6 Normed vector spaces 87  
3.3.7 From vector spaces to Hilbert spaces 90  
3.3.8 Bounded operators 91  
3.3.9 Adjoint operators 92  
3.3.10 Unitary and orthogonal operators 93  
3.3.11 Self-adjoint operators 96  
3.3.12 Compact operators 98  
3.3.13 Positive definite operators 99

3.4 Topological vector spaces 102  
3.5 Generalized eigendecomposition of matrices 103  
3.6 Singular value decomposition 103  
3.6.1 Geometric interpretation of SVD 104  
3.6.2 Low-rank approximation 106

3.7 Principal component analysis 108  
3.7.1 PCA algorithm 111  
3.7.2 PCA theory 114  
3.7.3 Scree plot 116  
3.7.4 Biplot 117
## Contents

4 Differential geometry of curves and surfaces 120

4.1 Local theory of curves 121
4.1.1 Curves and their tangents 121
4.1.2 Arc-length 124
4.1.3 Length of curves 125
4.1.4 Curvature of plane curves 126
4.1.5 Curvature and torsion of space curves 128
4.1.6 Fundamental theorem of curves 132
4.1.7 Implicit representation of curves in the plane 133

4.2 Local theory of surfaces 134
4.2.1 Parametric representation of surfaces 134
4.2.2 Tangent plane 137
4.2.3 Vector fields on surfaces 139
4.2.4 Gauss map 140
4.2.5 First fundamental form 140
4.2.6 Isometric surfaces 143
4.2.7 Geodesics 144
4.2.8 Area of a surface 148
4.2.9 Second fundamental form 148
4.2.10 Gaussian, mean, and principal curvatures 150
4.2.11 Orientability 157

4.3 Image segmentation using curve evolution 158

5 Geometric and differential topology of manifolds 168

5.1 Manifolds 169
5.1.1 Topological manifolds 169
5.1.2 Manifold with boundary 170
5.1.3 Smooth manifolds and smooth maps 171
5.1.4 Vector fields 174
5.1.5 Orientability 175
5.1.6 Pushforward and pullback 175
5.1.7 Whitney embedding theorem 177
5.1.8 Connections on manifolds 178
5.1.9 Quotient topology 179

5.2 Riemannian manifolds 179
5.2.1 Riemannian metric 181
5.2.2 Riemannian manifold 182
5.2.3 Area of a manifold 182
5.2.4 Laplace–Beltrami operator 182
5.2.5 Isometric manifolds 183

5.3 Graphs and topology 184
5.3.1 Triangular mesh representation 185
5.3.2 Topological invariants 186
Contents

5.3.3 Introduction to spectral graph theory 188
5.3.4 Introduction to spectral geometry 199

5.4 Introduction to Morse theory 203
5.4.1 Morse function 203
5.4.2 Level sets around Morse points 204
5.4.3 Handle decomposition 205
5.4.4 Reeb graph 205
5.4.5 Distance function 206

5.5 Applications 209
5.5.1 Shading problem 209
5.5.2 Morse-theoretic analysis of 3D shapes 212
5.5.3 Curve evolution on a manifold 218
5.5.4 Spectral graph wavelets for deformable 3D shape retrieval 220

6 Computational algebraic topology 238

6.1 Topological characterization by function evaluation 239
6.1.1 Morse function: a topological perspective 240
6.1.2 Almost all images are Morse functions 241
6.1.3 Topological equivalence of images 242

6.2 Discrete Morse theory: introduction 243
6.2.1 A simplex-based space 243
6.2.2 Critical simplices 246
6.2.3 A gradient vector field on a simplicial complex 247
6.2.4 Optimizing a discrete Morse function 248

6.3 An algebraic approach to topological analysis 249
6.3.1 Mapping-based equivalence of spaces 250
6.3.2 Simplicial homology 251
6.3.3 Singular homology 256
6.3.4 Homology-based topology characterization 258

6.4 Computational aspects of homology 259
6.4.1 Computing homology 260
6.4.2 Sensor networks: the coverage problem 261
6.4.3 Hole detection 264
6.4.4 Social networks 269

References 274
Index 282