1 Foundations of Newtonian gravity

The central theme of this book is gravitation in its weak-field aspects, as described within the framework of Einstein's general theory of relativity. Because Newtonian gravity is recovered in the limit of very weak fields, it is an appropriate entry point into our discussion of weak-field gravitation. Newtonian gravity, therefore, will occupy us within this chapter, as well as the following two chapters.

There are, of course, many compelling reasons to begin a study of gravitation with a thorough review of the Newtonian theory; some of these are reviewed below in Sec. 1.1. The reason that compels us most of all is that although there is a vast literature on Newtonian gravity – a literature that has accumulated over more than 300 years – much of it is framed in old mathematical language that renders it virtually impenetrable to present-day students. This is quite unlike the situation encountered in current presentations of Maxwell's electrodynamics, which, thanks to books such as Jackson's influential text (1998), are thoroughly modern. One of our main goals, therefore, is to submit the classical literature on Newtonian gravity to a Jacksonian treatment, to modernize it so as to make it accessible to present-day students. And what a payoff is awaiting these students! As we shall see in Chapters 2 and 3, Newtonian gravity is most generous in its consequences, delivering a whole variety of fascinating phenomena.

Another reason that compels us to review the Newtonian formulation of the laws of gravitation is that much of this material will be recycled and put to good use in later chapters of this book, in which we examine relativistic aspects of gravitation. Newtonian gravity, in this context, is a necessary warm-up exercise on the path to general relativity.

In this chapter we describe the foundations of the Newtonian theory, and leave the exploration of consequences to Chapters 2 and 3. We begin in Sec. 1.1 with a discussion of the domain of validity of the Newtonian theory. The main equations are displayed in Sec. 1.2 and derived systematically in Secs. 1.3 and 1.4. The gravitational fields of spherical and nearly spherical bodies are described in Sec. 1.5, and in Sec. 1.6 we derive the equations that govern the center-of-mass motion of extended fluid bodies.

Gravitation rules the world, and before Einstein ruled gravitation, Newton was its king. In this chapter and the following two we pay tribute to the king.

1.1 Newtonian gravity

The gravitational theory of Newton is an extremely good representation of gravity for a host of situations of practical and astronomical interest. It accurately describes the structure

Table 1.1 Values of ε for representative gravitating systems.

Earth's orbit around the Sun	10^{-8}
Solar system's orbit around the galaxy	10^{-6}
Surface of the Sun	10^{-5}
Surface of a white dwarf	10^{-4}
Surface of a neutron star	0.1
Event horizon of a black hole	~ 1

of the Earth and the tides raised on it by the Moon and Sun. It gives a detailed account of the orbital motion of the Moon around the Earth, and of the planets around the Sun. To be sure, it is now well established that the Newtonian theory is not an exact description of the laws of gravitation. As early as the middle of the 19th century, observations of the orbit of Mercury revealed a discrepancy with the prediction of Newtonian gravity. This famous discrepancy in the rate of advance of Mercury's perihelion was resolved by taking into account the relativistic corrections of Einstein's theory of gravity. The high precision of modern measuring devices has made it possible to detect relativistic effects in the lunar orbit, and has made it necessary to take relativity into account in precise tracking of planets and spacecraft, as well as in accurate measurements of the positions of stars using techniques such as Very Long Baseline Radio Interferometry (VLBI). Even such mundane daily activities as using the Global Positioning System (GPS) to navigate your car in a strange city require incorporation of special and general relativistic effects on the observed rates of the orbiting atomic clocks that regulate the GPS network. But apart from these specialized situations requiring very high precision, Newtonian gravity rules the solar system.

Newtonian gravity also rules for the overwhelming majority of stars in the universe. The structure and evolution of the Sun and other main-sequence stars can be completely and accurately treated using Newtonian gravity. Only for extremely compact stellar objects, such as neutron stars and, of course, black holes, is general relativity important. Newtonian gravity is also perfectly capable of handling the structure and evolution of galaxies and clusters of galaxies. Even the evolution of the largest structures in the universe, the great galactic clusters, sheets and voids, whose formation is dominated by the gravitational influence of dark matter, are frequently modelled using numerical simulations based on Newton's theory, albeit with the overall expansion of the universe playing a significant role.

Generally speaking, the criterion that we use to decide whether to employ Newtonian gravity or general relativity is the magnitude of a quantity called the "relativistic correction factor" ε:

$$\varepsilon \sim \frac{GM}{c^2 r} \sim \frac{v^2}{c^2}, \tag{1.1}$$

where G is the Newtonian gravitational constant, c is the speed of light, and where M, r, and v represent the characteristic mass, separation or size, and velocity of the system under consideration. The smaller this factor, the better is Newtonian gravity as an approximation. Table 1.1 shows representative values of ε for various systems.

Context is everything, of course. It is now accepted that general relativity, not Newtonian theory, is the "correct" classical theory of gravitation. But in the appropriate context, Newton's theory may be completely adequate to do the job at hand to the precision required. For example, Table 1.1 implies that a description of planetary motion around the Sun, at a level of accuracy limited to (say) one part in a million, can safely be based on the Newtonian laws. The Newtonian theory can also be exploited to calculate the internal structure of white dwarfs, provided that one is content with a level of accuracy limited to one part in one thousand. For more compact objects, such as neutron stars and black holes, Newtonian theory is wholly inadequate.

1.2 Equations of Newtonian gravity

Most undergraduate textbooks begin their treatment of Newtonian gravity with Newton's second law and the inverse-square law of gravitation:

$$m_I \boldsymbol{a} = \boldsymbol{F}\,, \tag{1.2a}$$

$$\boldsymbol{F} = -\frac{G m_G M}{r^2}\,\boldsymbol{n}\,. \tag{1.2b}$$

In the first equation, $\boldsymbol{F}$ is the force acting on a body of inertial mass m_I situated at position $\boldsymbol{r}(t)$, and $\boldsymbol{a} = d^2\boldsymbol{r}/dt^2$ is its acceleration. In the second equation, the force is assumed to be gravitational in nature, and to originate from a gravitating mass situated at the origin of the coordinate system. The force law involves m_G, the passive gravitational mass of the first body at $\boldsymbol{r}$, while M is the active gravitational mass of the second body. The quantity G is Newton's constant of gravitation, equal to $6.6738 \pm 0.0008 \times 10^{-11}\ \mathrm{m^2\,kg^{-1}\,s^{-2}}$. The force is attractive, it varies inversely with the square of the distance $r := |\boldsymbol{r}| = (x^2 + y^2 + z^2)^{1/2}$, and it points in the direction opposite to the unit vector $\boldsymbol{n} := \boldsymbol{r}/r$. An alternative form of the force law is obtained by writing it as the gradient of a potential $U = GM/r$, so that

$$\boldsymbol{F} = m_G \boldsymbol{\nabla} U\,. \tag{1.3}$$

This *Newtonian potential* will play a central role in virtually all chapters of this book.

If the inertial and passive gravitational masses of the body are equal to each other, $m_I = m_G$, then the acceleration of the body is given by $\boldsymbol{a} = \boldsymbol{\nabla} U$, and its magnitude is $a = GM/r^2$. Under this condition the acceleration is independent of the mass of the body. This statement is known as the *weak equivalence principle* (WEP), and it was a central element in Einstein's thinking on his way to the concepts of curved spacetime and general relativity. Although Newton did not explicitly use our formulation in terms of inertial and passive masses, he was well aware of the significance of their equality. In fact, he regarded this equality as so fundamental that he opened his treatise *Philosophiae Naturalis Principia Mathematica* with a discussion of it; he even alluded to his own experiments showing that the periods of pendulums were independent of the mass and type of material suspended, which establishes the equality of inertial and passive masses (he referred to them as the "quantity" and "weight" of bodies, respectively). Twentieth-century experiments have shown that the two types of mass are equal to parts in 10^{13} for a wide variety of materials (see Box 1.1).

Box 1.1 Tests of the weak equivalence principle

A useful way to discuss experimental tests of the weak equivalence principle is to parameterize the way it could be violated. In one parameterization, we imagine that a body is made up of atoms, and that the inertial mass m_I of an atom consists of the sum of all the mass and energy contributions of its constituents. But we suppose that the different forms of energy may contribute differently to the gravitational mass m_G than they do to m_I. One way to express this is to write

$$m_G = m_I(1 + \eta)\,,$$

where η is a dimensionless parameter that measures the difference. Because different forms of energy arising from the relevant subatomic interactions (such as electromagnetic and nuclear interactions) contribute different amounts to the total, depending on atomic structure, η could depend on the type of atom. For example, electrostatic energy of the nuclear protons contributes a much larger fraction of the total mass for high-*Z* atoms than for low-*Z* atoms.

Using this parameterization, we find from Eq. (1.2) that the acceleration of the body is given by

$$\boldsymbol{a} = -\frac{m_G}{m_I}\frac{GM}{r^2}\,\boldsymbol{n} = -(1+\eta)\,\frac{GM}{r^2}\,\boldsymbol{n}\,.$$

The difference in acceleration between two materials of different composition will then be given by

$$\Delta\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2 = -(\eta_1 - \eta_2)\,\frac{GM}{r^2}\,\boldsymbol{n}\,.$$

One way to place a bound on $\eta_1 - \eta_2$ is to drop two different objects in the Earth's gravitational field ($g = GM/r^2 \approx 9.8\,\mathrm{m\,s^{-2}}$), and compare their accelerations, or how long they take to fall. Although legend has it that Galileo Galilei verified the equivalence principle by dropping objects off the Leaning Tower of Pisa around 1590, in fact experiments like this had already been performed and were well known to Galileo; if he did indeed drop things off the Tower, he may simply have been performing a kind of classroom demonstration of an established fact for his students. Unfortunately, the "Galileo approach" is plagued by experimental errors, such as the difficulty of releasing the objects at exactly the same time, by the effects of air drag, and by the short time available for timing the drop.

A better approach is to balance the gravitational force (which depends on m_G) by a support force (which depends on m_I); the classic model is the pendulum experiments performed by Newton and reported in his *Principia*. The period of the pendulum depends on m_G/m_I, g, and the length of the pendulum. These experiments are also troubled by air drag, by errors in measuring or controlling the length of the pendulum, and by errors in timing the swing.

The best approach for laboratory tests was pioneered by Baron Roland von Eötvös, a Hungarian geophysicist working around the turn of the 20th century. He developed the *torsion balance*, schematically consisting of a rod suspended by a wire near its mid-point, with objects consisting of different materials attached at each end.

The point where the wire is attached to achieve a horizontal balance depends only on the gravitational masses of the two objects, so this configuration does not tell us anything. But if an additional gravitational force can be applied in a direction perpendicular to the supporting wire, and if there is a difference in m_G/m_I for the two bodies, then the rod will rotate in one direction or the other and the wire will twist until the restoring force of the twisted wire halts the rotation. There is no effect when m_G/m_I is the same for the two bodies. The additional force could be provided by a nearby massive body in the laboratory, a nearby mountain, the Sun, or the galaxy. Eötvös realized that, because of the centrifugal force produced by the rotation of the Earth, the wire hangs not exactly vertically, but is tilted slightly toward the south; at the latitude of Budapest, Hungary, the angle of tilt is about 0.1 degrees. Thus the gravitational acceleration of the Earth has a small component, about $g/400$, perpendicular to the wire, in a northerly direction. By slowly rotating the whole apparatus carefully about the vertical direction, Eötvös could compare the twist in two opposite orientations of the rod, and thereby eliminate a number of sources of error.

Eötvös found no measurable twist, within his experimental errors, for many different combinations of materials, and he was able to place an upper limit of $|\eta_1 - \eta_2| < 3 \times 10^{-9}$, corresponding to a limit on any difference in acceleration of the order of $7 \times 10^{-11}\ \mathrm{m\,s^{-2}}$. Even though the driving acceleration is only a tiny fraction of g, there is an enormous gain in sensitivity to tiny accelerations, mainly because the apparatus is almost static and can be observed for long periods of time. Torsion balance experiments were improved by Robert Dicke in Princeton and Vladimir Braginsky in Moscow during the 1960s and 1970s, and again during the 1980s as part of a search for a hypothetical "fifth" force (no evidence for such a force was found). The most recent experiments, performed notably by the "Eöt-Wash" group at the University of Washington, Seattle, have reached precisions of a few parts in 10^{13}; these experiments used the Sun or the galaxy as the source of gravity.

All these experiments exploit only a tiny fraction of the available acceleration. The only way to make full use of g while maintaining high sensitivity to acceleration differences is to design a "perpetual" Galileo drop experiment, namely by putting the different bodies in orbit around the Earth. Various satellite tests of the equivalence principle are in preparation, with the goal of reaching sensitivities ranging from 10^{-15} to 10^{-18}. Such experiments come with a high monetary cost: compared to laboratory experiments, space experiments are extraordinarily expensive.

Another test of the equivalence principle was carried out using the Earth–Moon system. The two bodies have slightly different compositions, with the Earth dominated by its iron–nickel core, and the Moon dominated by silicates. If there were a violation of the equivalence principle, the two bodies would fall with different accelerations toward the Sun, and this would have an effect on the Earth–Moon orbit. Lunar laser ranging is a technique of bouncing laser beams off reflectors placed on the lunar surface during the American and Soviet lunar landing programs of the 1970s, and it has reached the capability of measuring the Earth–Moon distance at the sub-centimeter level. No evidence for such a perturbation in the Earth–Moon distance has been found, so that the Earth and the Moon obey the equivalence principle to a few parts in 10^{13}. We describe the laser ranging measurements of the Moon in more detail in Box 13.2.

The weak equivalence principle is one of the most important foundational elements of relativistic theories of gravity. We will return to it in Chapter 5, on our way to general relativity.

We shall assume that the weak equivalence principle holds perfectly, and make this an axiom of Newtonian gravity. We shall return to this principle in Chapter 5 and present it as an essential foundational element of general relativity, and we shall return to it again in Chapter 13 – in a different version known as the *strong equivalence principle* – and present it as a highly non-trivial property of massive, self-gravitating bodies in general relativity.

The weak equivalence principle allows us to rewrite Eqs. (1.2) in the form of an equation of motion for the body at $\boldsymbol{r}(t)$, and a field equation for the potential U:

$$\boldsymbol{a} = \nabla U \,, \tag{1.4a}$$

$$U = GM/r \,. \tag{1.4b}$$

These equations are limited in scope, and they do not yet form the final set of equations that will be adopted as the foundations of Newtonian gravity. Their limitation has to do with the fact that they apply to a point mass situated at $\boldsymbol{r}(t)$ being subjected to the gravitational force produced by another point mass situated at the origin of the coordinate system. We are interested in much more general situations. First, we wish to consider the motion of extended bodies made up of continuous matter (solid, fluid, or gas), allowing the bodies to be of arbitrary size, shape, and constitution, and possibly to evolve in time according to their own internal dynamics. Second, we wish to consider an arbitrary number of such bodies, and to put them all on an equal footing; each body will be subjected to the gravity of the remaining bodies, and each will move in response to this interaction.

These goals can be achieved by generalizing the primitive Eqs. (1.4) to a form that applies to a continuous distribution of matter. We shall perform this generalization in Secs. 1.3 and 1.4, but to complete the discussion of this section, we choose to immediately list and describe the resulting equations.

Our formulation of the fundamental equations of Newtonian gravity relies on a fluid description of matter, in which the matter distribution is characterized by a mass-density field $\rho(t, \boldsymbol{x})$, a pressure field $p(t, \boldsymbol{x})$, and a velocity field $\boldsymbol{v}(t, \boldsymbol{x})$; these quantities depend on time t and position $\boldsymbol{x}$ within the fluid. Our formulation relies also on the Newtonian potential $U(t, \boldsymbol{x})$, which also depends on time and position, and which provides a description of the gravitational field. The equations that govern the behavior of the matter are the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \,, \tag{1.5}$$

which expresses the conservation of mass, and *Euler's equation*,

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \nabla U - \nabla p \,, \tag{1.6}$$

which is the generalization of Eq. (1.4a) to continuous matter; here

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \tag{1.7}$$

is the convective time derivative associated with the motion of fluid elements. The equation that governs the behavior of the gravitational field is *Poisson's equation*

$$\nabla^2 U = -4\pi G\rho \,, \tag{1.8}$$

where

$$\nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.9}$$

is the familiar Laplacian operator; Poisson's equation (known after its originator Siméon Denis Poisson, who unfortunately is not related to either author of this book) is the generalization of Eq. (1.4b) to continuous matter.

As was stated previously, these equations will be properly introduced in the following two sections. To complete the formulation of the theory we must impose a relationship between the pressure and the density of the fluid. This relationship, known as the *equation of state*, takes the general form of

$$p = p(\rho, T, \cdots), \tag{1.10}$$

in which the pressure is expressed as a function of the density, temperature, and possibly other relevant variables such as chemical composition. The equation of state encodes information about the microphysics that governs the fluid, and this information must be provided as an input in most applications of the theory.

A complete description of a physical situation involving gravity and a distribution of matter can be obtained by integrating Eqs. (1.5), (1.6), and (1.8) simultaneously and self-consistently. The solutions must be subjected to suitable boundary conditions, which will be part of the specification of the problem. All of Newtonian gravity is contained in these equations, and all associated phenomena follow as consequences of these equations.

1.3 Newtonian field equation

In this section we examine the equations that govern the behavior of the gravitational field, and show how Eq. (1.8) is an appropriate generalization of the more primitive form of Eq. (1.4b).

We recall that the relation $U = GM/r$ applies to a point body of active gravitational mass M situated at the origin of the coordinate system. Suppose that we are given an arbitrary number N of point bodies, and that we assign to each one a label $A = 1, 2, \cdots, N$. The mass and position of each body are then denoted M_A and $\boldsymbol{r}_A(t)$, respectively. If we *assume* that the total Newtonian potential U is a linear superposition of the individual potentials U_A created by each body, we have that the potential at position $\boldsymbol{x}$ is given by

$$U = \sum_A U_A = G \sum_A \frac{M_A}{|\boldsymbol{x} - \boldsymbol{r}_A|}. \tag{1.11}$$

The generalization of this relation to a continuous distribution of matter is straightforward. We convert the discrete sum $\sum_A M_A$ to a continuous integral $\int d^3x' \rho(t, \boldsymbol{x}')$, and we replace the discrete positions $\boldsymbol{r}_A$ with the continuous integration variable $\boldsymbol{x}'$. The result is

$$U(t, \boldsymbol{x}) = G \int \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x', \tag{1.12}$$

one of the key defining equations for the Newtonian potential. The integral can be evaluated as soon as the density field $\rho(t, \boldsymbol{x}')$ is specified, regardless of whether ρ is a proper solution to the remaining fluid equations. As such, Eq. (1.12) gives U as a *functional* of an arbitrary function ρ. The potential, however, will be physically meaningful only when ρ itself is physically meaningful, which means that it must be a proper solution to the continuity and Euler equations.

The integral equation (1.12) can easily be transformed into a differential equation for the Newtonian potential U. The transformation relies on the identity

$$\nabla^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.13}$$

in which $\delta(\boldsymbol{x}-\boldsymbol{x}') := \delta(x-x')\delta(y-y')\delta(z-z')$ is a three-dimensional delta function defined by the properties

$$\delta(\boldsymbol{x}-\boldsymbol{x}') = 0 \qquad \text{when } \boldsymbol{x} \neq \boldsymbol{x}', \tag{1.14a}$$

$$f(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}') = f(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}') \qquad \text{for any smooth function } f(\boldsymbol{x}), \tag{1.14b}$$

$$\int \delta(\boldsymbol{x}-\boldsymbol{x}')\, d^3x' = 1 \qquad \text{for any domain of integration that encloses } \boldsymbol{x}. \tag{1.14c}$$

These properties further imply that $\delta(\boldsymbol{x}'-\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{x}')$. The identity of Eq. (1.13) is derived in Box 1.2. If we apply the Laplacian operator on both sides of Eq. (1.12) and exchange the operations of integration and differentiation on the right-hand side, we obtain

$$\begin{aligned}\nabla^2 U &= G\int \rho(t, \boldsymbol{x}')\nabla^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' \\ &= -4\pi G\int \rho(t, \boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')\, d^3x' \\ &= -4\pi G\rho(t, \boldsymbol{x});\end{aligned}$$

the identity was used in the second step, and the properties of the delta function displayed in Eq. (1.14) allowed us to evaluate the integral. The end result is Poisson's equation,

$$\nabla^2 U = -4\pi G\rho, \tag{1.15}$$

whose formulation was anticipated in Eq. (1.8).

It is possible to proceed in the opposite direction, and show that Eq. (1.12) provides a solution to Poisson's equation (1.15). A powerful tool in the integration of differential equations is the *Green's function* $G(\boldsymbol{x}, \boldsymbol{x}')$, a function of a field point $\boldsymbol{x}$ and a source point $\boldsymbol{x}'$. In the specific context of Poisson's equation, the Green's function is required to be a solution to

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.16}$$

which is recognized as a specific case of the general differential equation, corresponding to a point mass situated at $\boldsymbol{x}'$. Armed with such an object, a formal solution to Eq. (1.15) can be expressed as

$$U(t, \boldsymbol{x}) = G\int G(\boldsymbol{x}, \boldsymbol{x}')\rho(t, \boldsymbol{x}')\, d^3x'; \tag{1.17}$$

Box 1.2 Proof that $\nabla^2|\boldsymbol{x}-\boldsymbol{x}'|^{-1} = -4\pi\,\delta(\boldsymbol{x}-\boldsymbol{x}')$

To simplify the proof of Eq. (1.13) we set $\boldsymbol{x}' = \boldsymbol{0}$ without loss of generality; this can always be achieved by a translation of the coordinate system. This gives rise to the simpler equation

$$\nabla^2 r^{-1} = -4\pi\,\delta(\boldsymbol{x}), \tag{1}$$

with $r := |\boldsymbol{x}|$.

We first show that $\nabla^2 r^{-1} = 0$ whenever $\boldsymbol{x} \neq \boldsymbol{0}$. Derivatives of r^{-1} can be evaluated with the help of the identities

$$\frac{\partial r}{\partial x^j} = n_j, \qquad \frac{\partial n_j}{\partial x^k} = \frac{\partial n_k}{\partial x^j} = \frac{1}{r}\big(\delta_{jk} - n_j n_k\big),$$

where $x^j := (x, y, z)$ is a component notation for the vector $\boldsymbol{x}$, $n^j := x^j/r$, and δ_{jk} is the Kronecker delta, equal to one when $j = k$ and zero otherwise. These equations hold provided that $r \neq 0$. According to this we have that

$$\frac{\partial}{\partial x^j} r^{-1} = -\frac{1}{r^2} n_j$$

and

$$\frac{\partial^2}{\partial x^j \partial x^k} r^{-1} = \frac{1}{r^3}\big(3n_j n_k - \delta_{jk}\big).$$

Because $\boldsymbol{n}$ is a unit vector, it follows that $\nabla^2 r^{-1} = 0$ whenever $r \neq 0$.

To handle the special case $r = 0$ we introduce the vector $\boldsymbol{j} := \boldsymbol{\nabla} r^{-1}$ and write the left-hand side of Eq. (1) as $\boldsymbol{\nabla}\cdot\boldsymbol{j}$. Integrating this over a volume V bounded by a spherical surface S of radius η, we obtain

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\, d^3x = \oint_S \boldsymbol{j}\cdot d\boldsymbol{S}$$

by virtue of Gauss's theorem. Here $d\boldsymbol{S}$ is an outward-directed surface element on S, which can be expressed as $d\boldsymbol{S} = \boldsymbol{n}\eta^2\, d\Omega$, with $d\Omega$ denoting an element of solid angle centered at $\boldsymbol{n}$. The vector $\boldsymbol{j}$ is equal to $-\eta^{-2}\boldsymbol{n}$ on S, and evaluating the surface integral returns -4π.

Because $\nabla^2 r^{-1}$ vanishes when $\boldsymbol{x} \neq 0$ and integrates to -4π whenever the integration domain encloses $\boldsymbol{x} = \boldsymbol{0}$, we conclude that it is distributionally equal to $-4\pi\,\delta(\boldsymbol{x})$. The proof is complete.

the steps involved in establishing that this U is indeed a solution to Poisson's equation are identical to those that previously led us to Eq. (1.15) from Eq. (1.12). The difference is that in the earlier derivation the identity of the Green's function was already known. In the approach described here, the result follows simply by virtue of Eq. (1.16). It is not difficult, of course, to identify the Green's function: comparison with Eq. (1.13) allows us to write

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{1.18}$$

Not surprisingly, the Green's function represents the potential of a point mass situated at $\boldsymbol{x}'$.

1.4 Equations of hydrodynamics

In this section we develop the foundations for the equations of hydrodynamics, as displayed previously in Eqs. (1.5) and (1.6).

1.4.1 Motion of fluid elements

Definition of fluid element

We begin by describing any material body as being made up of *fluid elements*, volumes of matter that are very small compared to the size of the body, but very large compared to the inter-molecular distance, so that the element contains a macroscopic number of molecules. The fluid description of matter is a coarse-grained one in which the molecular fluctuations are smoothed over, and the fluid element is meant to represent a local average of the matter contained within. The coarse-graining could be described in great detail, for example, by introducing a microscopic density $\eta(t, \boldsymbol{x})$ that fluctuates wildly on the molecular scale, as well as smoothing function $w(|\boldsymbol{x} - \boldsymbol{x}'|)$ that varies over a much larger scale; the macroscopic density would then be defined as $\rho(t, \boldsymbol{x}) = \int \eta(t, \boldsymbol{x}') w(|\boldsymbol{x} - \boldsymbol{x}'|)\, d^3x'$. We will not go into such depth here, and keep the discussion at an intuitive, elementary level.

Each fluid element can be characterized by a mass density ρ (the mass of the element divided by its volume), a pressure p (the normal force per unit area acting on the surface of the element), and a velocity $\boldsymbol{v}$ (the average velocity of the molecules in the element). Other variables, such as viscosity, temperature, entropy, mean atomic weight, opacity, and so on, can also be introduced (some of these appear in Sec. 1.4.2). Apart from the velocity, all fluid variables are assumed to be measured by an observer who is momentarily at rest with respect to the fluid element. This description is adequate in a Newtonian setting, but it will have to be refined later, when we transition to the relativistic setting of Chapters 4 and 5.

Perhaps the most important aspect of a fluid element is that it keeps its contents intact as it moves within the fluid. During the motion the element may alter its shape and even its volume, but it will always contain the same collection of molecules; by definition no molecule is allowed to enter or leave the element. (It may be helpful to think of the molecules as being tagged, and of the fluid element as a bag that contains the tagged molecules.) A very important consequence of this property is that the total mass contained in a fluid element will never change; it is a constant of the element's motion.

Euler equation

We now apply Newton's laws to a selected fluid element of volume $\mathcal{V}$. The mass of the element is $\rho\mathcal{V}$, and from Newton's second law we have that

$$(\rho\mathcal{V})\boldsymbol{a} = \boldsymbol{F}\,, \tag{1.19}$$