

Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

## A COURSE IN MATHEMATICAL ANALYSIS

### Volume III: Complex Analysis, Measure and Integration

The three volumes of *A Course in Mathematical Analysis* provide a full and detailed account of all those elements of real and complex analysis that an undergraduate mathematics student can expect to encounter in the first two or three years of study. Containing hundreds of exercises, examples and applications, these books will become an invaluable resource for both students and instructors.

Volume I focuses on the analysis of real-valued functions of a real variable. Volume II goes on to consider metric and topological spaces, and functions of a vector variable, and includes an introduction to the theory of manifolds in Euclidean space. This third volume develops the classical theory of functions of a complex variable. It carefully establishes the properties of the complex plane, including a proof of the Jordan curve theorem. Lebesgue measure is introduced, and is used as a model for other measure spaces, where the theory of integration is developed. The Radon–Nikodym theorem is proved, and the differentiation of measures is discussed.

D. J. H. GARLING is Emeritus Reader in Mathematical Analysis at the University of Cambridge and Fellow of St. John's College, Cambridge. He has fifty years' experience of teaching undergraduate students in most areas of pure mathematics, but particularly in analysis.

Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

---

Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

# A COURSE IN MATHEMATICAL ANALYSIS

Volume III  
Complex Analysis,  
Measure and Integration

D. J. H. GARLING

*Emeritus Reader in Mathematical Analysis,  
University of Cambridge, and*

*Fellow of St John's College, Cambridge*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

**CAMBRIDGE**  
UNIVERSITY PRESS

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of  
education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781107032040](http://www.cambridge.org/9781107032040)

© D. J. H. Garling 2014

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without the written  
permission of Cambridge University Press.

First published 2014

Printed in the United Kingdom by CPI Group Ltd, Croydon CR0 4YY

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication Data*

Garling, D. J. H.

Foundations and elementary real analysis / D. J. H. Garling.

pages cm. – (A course in mathematical analysis; volume 1)

Includes bibliographical references and index.

ISBN 978-1-107-03202-6 (hardback) – ISBN 978-1-107-61418-5 (paperback)

1. Mathematical analysis. I. Title.

QA300.G276 2013

515–dc23 2012044420

ISBN 978-1-107-03204-0 Hardback

ISBN 978-1-107-66330-5 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of  
URLs for external or third-party internet websites referred to in this publication,  
and does not guarantee that any content on such websites is, or will remain,  
accurate or appropriate.

## Contents

### Volume III

<i>Introduction</i>	<i>page</i> ix
<b>Part Five Complex analysis</b>	625
<b>20 Holomorphic functions and analytic functions</b>	627
20.1 Holomorphic functions	627
20.2 The Cauchy–Riemann equations	630
20.3 Analytic functions	635
20.4 The exponential, logarithmic and circular functions	641
20.5 Infinite products	645
20.6 The maximum modulus principle	646
<b>21 The topology of the complex plane</b>	650
21.1 Winding numbers	650
21.2 Homotopic closed paths	655
21.3 The Jordan curve theorem	661
21.4 Surrounding a compact connected set	667
21.5 Simply connected sets	670
<b>22 Complex integration</b>	674
22.1 Integration along a path	674
22.2 Approximating path integrals	680
22.3 Cauchy’s theorem	684
22.4 The Cauchy kernel	689
22.5 The winding number as an integral	690
22.6 Cauchy’s integral formula for circular and square paths	692
22.7 Simply connected domains	698
22.8 Liouville’s theorem	699

22.9	Cauchy's theorem revisited	700
22.10	Cycles; Cauchy's integral formula revisited	702
22.11	Functions defined inside a contour	704
22.12	The Schwarz reflection principle	705
<b>23</b>	<b>Zeros and singularities</b>	<b>708</b>
23.1	Zeros	708
23.2	Laurent series	710
23.3	Isolated singularities	713
23.4	Meromorphic functions and the complex sphere	718
23.5	The residue theorem	720
23.6	The principle of the argument	724
23.7	Locating zeros	730
<b>24</b>	<b>The calculus of residues</b>	<b>733</b>
24.1	Calculating residues	733
24.2	Integrals of the form $\int_0^{2\pi} f(\cos t, \sin t) dt$	734
24.3	Integrals of the form $\int_0^\infty f(x) dx$	736
24.4	Integrals of the form $\int_0^\infty x^\alpha f(x) dx$	742
24.5	Integrals of the form $\int_0^\infty f(x) dx$	745
<b>25</b>	<b>Conformal transformations</b>	<b>749</b>
25.1	Introduction	749
25.2	Univalent functions on $\mathbf{C}$	750
25.3	Univalent functions on the punctured plane $\mathbf{C}^*$	750
25.4	The Möbius group	751
25.5	The conformal automorphisms of $\mathbf{D}$	758
25.6	Some more conformal transformations	759
25.7	The space $\mathbf{H}(U)$ of holomorphic functions on a domain $U$	763
25.8	The Riemann mapping theorem	765
<b>26</b>	<b>Applications</b>	<b>768</b>
26.1	Jensen's formula	768
26.2	The function $\pi \cot \pi z$	770
26.3	The functions $\pi \operatorname{cosec} \pi z$	772
26.4	Infinite products	775
26.5	*Euler's product formula*	778
26.6	Weierstrass products	783
26.7	The gamma function revisited	790
26.8	Bernoulli numbers, and the evaluation of $\zeta(2k)$	794
26.9	The Riemann zeta function revisited	797

*Contents*

vii

<b>Part Six Measure and Integration</b>	801
<b>27 Lebesgue measure on <math>\mathbf{R}</math></b>	803
27.1 Introduction	803
27.2 The size of open sets, and of closed sets	804
27.3 Inner and outer measure	808
27.4 Lebesgue measurable sets	810
27.5 Lebesgue measure on $\mathbf{R}$	812
27.6 A non-measurable set	814
<b>28 Measurable spaces and measurable functions</b>	817
28.1 Some collections of sets	817
28.2 Borel sets	820
28.3 Measurable real-valued functions	822
28.4 Measure spaces	825
28.5 Null sets and Borel sets	829
28.6 Almost sure convergence	830
<b>29 Integration</b>	834
29.1 Integrating non-negative functions	834
29.2 Integrable functions	839
29.3 Changing measures and changing variables	846
29.4 Convergence in measure	848
29.5 The spaces $L_{\mathbf{R}}^1(X, \Sigma, \mu)$ and $L_{\mathbf{C}}^1(X, \Sigma, \mu)$	854
29.6 The spaces $L_{\mathbf{R}}^p(X, \Sigma, \mu)$ and $L_{\mathbf{C}}^p(X, \Sigma, \mu)$ , for $0 < p < \infty$	856
29.7 The spaces $L_{\mathbf{R}}^\infty(X, \Sigma, \mu)$ and $L_{\mathbf{C}}^\infty(X, \Sigma, \mu)$	863
<b>30 Constructing measures</b>	865
30.1 Outer measures	865
30.2 Caratheodory's extension theorem	868
30.3 Uniqueness	871
30.4 Product measures	873
30.5 Borel measures on $\mathbf{R}$ , $\mathbf{I}$	880
<b>31 Signed measures and complex measures</b>	884
31.1 Signed measures	884
31.2 Complex measures	889
31.3 Functions of bounded variation	891
<b>32 Measures on metric spaces</b>	896
32.1 Borel measures on metric spaces	896
32.2 Tight measures	898
32.3 Radon measures	900

Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

viii

*Contents*

<b>33 Differentiation</b>	903
33.1 The Lebesgue decomposition theorem	903
33.2 Sublinear mappings	906
33.3 The Lebesgue differentiation theorem	908
33.4 Borel measures on $\mathbf{R}$ , II	912
<b>34 Applications</b>	915
34.1 Bernstein polynomials	915
34.2 The dual space of $L^p_{\mathbf{C}}(X, \Sigma, \mu)$ , for $1 \leq p < \infty$	918
34.3 Convolution	919
34.4 Fourier series revisited	924
34.5 The Poisson kernel	927
34.6 Boundary behaviour of harmonic functions	934
<i>Index</i>	936
<i>Contents for Volume I</i>	940
<i>Contents for Volume II</i>	943



Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

## Introduction

This book is the third and final volume of a full and detailed course in the elements of real and complex analysis that mathematical undergraduates may expect to meet. Indeed, I have based it on those parts of analysis that undergraduates at Cambridge University meet, or used to meet, in their first two years. I have however found it desirable to go rather further in certain places, in order to give a rounded account of the material.

In Part Five, we develop the theory of functions of a complex variable. To begin with, we consider holomorphic functions (functions which are complex-differentiable) and analytic functions (functions which can be defined by power series), and the results seem similar to those of real case. Things change when path-integrals are introduced. To use these, a good understanding of the topology of the plane is needed. We give a careful account of this, including a proof of the Jordan curve theorem (every simple closed curve has an inside and an outside). With this in place, various forms of Cauchy's theorem and Cauchy's integral formula are proved. These lead on to many magical results. Chapter 25 is geometric. A single-valued holomorphic function is conformal (that is, it preserves angles and orientations). We consider the problem of mapping one domain conformally onto another, and end by proving the celebrated Riemann mapping theorem, which says that if  $U$  and  $V$  are domains in the complex plane which are proper subsets of the plane and are simply-connected (there are no holes) then there exists a conformal mapping of  $U$  onto  $V$ . In Chapter 26, we apply the theory that we have developed to various problems, some of which were first introduced in Volume I.

In Volume I, we developed properties of the Riemann integral. This is very satisfactory when we wish to integrate continuous or monotonic functions, and is a useful precursor for the complex path integrals that we consider in Part Five, but it has serious shortcomings. In Part Six, we introduce

Cambridge University Press

978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis,  
Measure and Integration

D. J. H. Garling

Frontmatter

[More information](#)

Lebesgue measure on the real line. Abstract measure theory is a large and important subject, but the topological properties of the real line make the construction of Lebesgue measure on the real line rather straightforward. With this example in place, we introduce the notion of a measure space, and the corresponding space of measurable functions. This then leads on easily to the theory of integration, and the space  $L^p$  of  $p$ -th power integrable functions. These results are used to construct Lebesgue measure in higher dimensions, using Fubini's theorem. Properties of the Hilbert space  $L^2$  are then used to give von Neumann's proof of the Radon–Nikodym theorem, and this is used to establish differentiability properties of measures and functions on  $\mathbf{R}^d$ . Almost all measures that arise in practice are defined on topological spaces, and we establish regularity properties, which show that such measures are rather well behaved. A final chapter uses the theory that we have established to obtain further results, largely concerning Fourier series (first considered in Volume I), and the boundary behaviour of harmonic functions on the unit disc.

The text includes plenty of exercises. Some are straightforward, some are searching, and some contain results needed later. All help develop an understanding of the theory: do them!

I am again extremely grateful to Zhuo Min 'Harold' Lim, who read the proofs and found many errors. Any remaining errors are mine alone. Corrections and further comments can be found on a web page on my personal home page at [www.dpmms.cam.ac.uk](http://www.dpmms.cam.ac.uk).