A COURSE IN MATHEMATICAL ANALYSIS

Volume III: Complex Analysis, Measure and Integration

The three volumes of *A Course in Mathematical Analysis* provide a full and detailed account of all those elements of real and complex analysis that an undergraduate mathematics student can expect to encounter in the first two or three years of study. Containing hundreds of exercises, examples and applications, these books will become an invaluable resource for both students and instructors.

Volume I focuses on the analysis of real-valued functions of a real variable. Volume II goes on to consider metric and topological spaces, and functions of a vector variable, and includes an introduction to the theory of manifolds in Euclidean space. This third volume develops the classical theory of functions of a complex variable. It carefully establishes the properties of the complex plane, including a proof of the Jordan curve theorem. Lebesgue measure is introduced, and is used as a model for other measure spaces, where the theory of integration is developed. The Radon–Nikodym theorem is proved, and the differentiation of measures is discussed.

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Volume III
Complex Analysis, Measure and Integration

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Contents

Volume III

Introduction

This book is the third and final volume of a full and detailed course in the elements of real and complex analysis that mathematical undergraduates may expect to meet. Indeed, I have based it on those parts of analysis that undergraduates at Cambridge University meet, or used to meet, in their first two years. I have however found it desirable to go rather further in certain places, in order to give a rounded account of the material.

In Part Five, we develop the theory of functions of a complex variable. To begin with, we consider holomorphic functions (functions which are complex-differentiable) and analytic functions (functions which can be defined by power series), and the results seem similar to those of real case. Things change when path-integrals are introduced. To use these, a good understanding of the topology of the plane is needed. We give a careful account of this, including a proof of the Jordan curve theorem (every simple closed curve has an inside and an outside). With this in place, various forms of Cauchy's theorem and Cauchy's integral formula are proved. These lead on to many magical results. Chapter 25 is geometric. A single-valued holomorphic function is conformal (that is, it preserves angles and orientations). We consider the problem of mapping one domain conformally onto another, and end by proving the celebrated Riemann mapping theorem, which says that if U and V are domains in the complex plane which are proper subsets of the plane and are simply-connected (there are no holes) then there exists a conformal mapping of U onto V. In Chapter 26, we apply the theory that we have developed to various problems, some of which were first introduced in Volume I.

In Volume I, we developed properties of the Riemann integral. This is very satisfactory when we wish to integrate continuous or monotonic functions, and is a useful precursor for the complex path integrals that we consider in Part Five, but it has serious shortcomings. In Part Six, we introduce

Lebesgue measure on the real line. Abstract measure theory is a large and important subject, but the topological properties of the real line make the construction of Lebesgue measure on the real line rather straightforward. With this example in place, we introduce the notion of a measure space, and the corresponding space of measurable functions. This then leads on easily to the theory of integration, and the space L^p of p-th power integrable functions. These results are used to construct Lebesgue measure in higher dimensions, using Fubini's theorem. Properties of the Hilbert space L^2 are then used to give von Neumann's proof of the Radon–Nikodym theorem, and this is used to establish differentiability properties of measures and functions on $\mathbf{R}^d$. Almost all measures that arise in practice are defined on topological spaces, and we establish regularity properties, which show that such measures are rather well behaved. A final chapter uses the theory that we have established to obtain further results, largely concerning Fourier series (first considered in Volume I), and the boundary behaviour of harmonic functions on the unit disc.

The text includes plenty of exercises. Some are straightforward, some are searching, and some contain results needed later. All help develop an understanding of the theory: do them!

I am again extremely grateful to Zhuo Min 'Harold' Lim, who read the proofs and found many errors. Any remaining errors are mine alone. Corrections and further comments can be found on a web page on my personal home page at `www.dpmms.cam.ac.uk`.