Part Five

Complex analysis

Cambridge University Press
978-1-107-03204-0 - A Course in Mathematical Analysis: Volume III: Complex Analysis, Measure and Integration
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Holomorphic functions and analytic functions

20.1 Holomorphic functions

Suppose that f is a continuous complex-valued function defined on an open subset U of the complex plane $\mathbf{C}$. Recall that the set U is the union of countably many connected components, each of which is an open subset of U (Volume II, Proposition 16.1.15 and Corollary 16.1.18). The behaviour of f on each component does not depend on its behaviour on the other components. For this reason, we restrict our attention to functions defined on a connected open subset of $\mathbf{C}$; such a set is called a *domain*.

We begin by considering differentiability: the definition is essentially the same as in the real case. Suppose that f is a complex-valued function on a domain U, and that $z \in U$. Then f is *differentiable* at z, with *derivative* $f'(z)$, if whenever $\epsilon > 0$ there exists $\delta > 0$ such that the open neighbourhood $N_\delta(z) = \{w : |w - z| < \delta\}$ of z is contained in U and such that if $0 < |w - z| < \delta$ then

$$\left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| < \epsilon.$$

In other words,

$$\frac{f(w) - f(z)}{w - z} \to f'(z) \text{ as } w \to z.$$

Thus if f is differentiable at z, then the derivative $f'(z)$ is uniquely determined. The derivative $f'(z)$ is also denoted by $\frac{df}{dz}(z)$.

Proposition 20.1.1 *Suppose that f is a complex-valued function on a domain U, that $N_\delta(z) \subseteq U$, and that $l \in \mathbf{C}$. The following statements are equivalent.*

(i) *f is differentiable at z, with derivative l.*
(ii) *There is a complex-valued function r on $N^*_\delta(0) = N_\delta(0) \setminus \{0\}$ such that*

$$f(z+w) = f(z) + lw + r(w) \text{ for } 0 < |w| < \delta$$

for which $r(w)/w \to 0$ as $w \to 0$.
(iii) *There is a complex-valued function s on $N_\delta(0)$ such that*

$$f(z+w) = f(z) + (l + s(w))w \text{ for } |w| < \delta$$

for which $s(0) = 0$ and s is continuous at 0.

If so, then f is continuous at z.

Proof This corresponds to Volume I, Proposition 7.1.1, and the easy proof is essentially the same. □

If f is differentiable at every point of U, then we say that f is *holomorphic* on U. If $U = \mathbf{C}$, then we say that f is an *entire* function. Although the form of the definition of differentiability that we have just given is exactly the same as the form of the definition in the real case, we shall see that holomorphic functions are very different from differentiable functions on an open interval of $\mathbf{R}$.

Example 20.1.2 Let $f(z) = 1/z$ for $z \in \mathbf{C} \setminus \{0\}$. Then f is holomorphic on $\mathbf{C} \setminus \{0\}$, with derivative $-1/z^2$.

For if $0 < |w| < |z|$, then $z + w \neq 0$ and

$$\frac{f(z+w) - f(z)}{w} - \frac{-1}{z^2} = \frac{z^2 - (z+w)z + w(z+w)}{wz^2(z+w)} = \frac{w}{z^2(z+w)} \to 0$$

as $w \to 0$.

Proposition 20.1.3 *Suppose that f and g are complex-valued functions defined on a domain U, and that f and g are differentiable at z. Suppose also that $\lambda, \mu \in \mathbf{C}$.*

(i) *$\lambda f + \mu g$ is differentiable at z, with derivative $\lambda f'(z) + \mu g'(z)$.*
(ii) *The product fg is differentiable at z, with derivative $f'(z)g(z) + f(z)g'(z)$.*

Proof An easy exercise for the reader. □

Theorem 20.1.4 (The chain rule) *Suppose that f is a complex-valued function defined on a domain U, that h is a complex-valued function defined* *on a domain V and that $f(U) \subseteq V$. Suppose that f is differentiable at z and that h is differentiable at $f(z)$. Then the composite function $h \circ f$ is differentiable at z, with derivative $h'(f(z)).f'(z)$.*

Proof There are two possibilities. First, there exists $\delta > 0$ such that $N_\delta(z) \subseteq U$ and $f(z+w) \neq f(z)$ for $0 < |w| < \delta$. If $0 < |w| < \delta$ then

$$\frac{h(f(z+w)) - h(f(z))}{w} = \left(\frac{h(f(z+w)) - h(f(z))}{f(z+w) - f(w)}\right) . \left(\frac{f(z+w) - f(z)}{w}\right).$$

Since f is continuous at z, $f(z+w) - f(z) \to 0$ as $w \to 0$, and so

$$\frac{h(f(z+w)) - h(f(z))}{f(z+w) - f(z)} \to h'(f(z)) \text{ as } w \to 0.$$

Since $(f(z+w) - f(z))/w \to f'(z)$ as $w \to 0$, the result follows.

Secondly, z is the limit point of a sequence $(z_n)_{n=1}^\infty$ in $U \setminus \{z\}$ for which $f(z_n) = f(z)$. In this case it follows that $f'(z) = 0$, and we must show that $(h \circ f)'(z) = 0$. We use Proposition 20.1.1. Let $b = f(z)$. There exist $\eta > 0$ such that $N_\eta(f(z)) \subseteq V$ and a function t on $N_\eta(0)$, with $t(0) = 0$, such that $h(b+k) = h(b) + (h'(b) + t(k))k$ for $k \in N_\eta(0)$ and such that t is continuous at 0. Similarly, there exist $\delta > 0$ such that $N_\delta(z) \subseteq U$ and a function s on $N_\delta(0)$, with $s(0) = 0$, such that $f(z+w) = b + s(w)w$ for $h \in N_\delta(0)$ and such that s is continuous at 0. Since f is continuous at z, we can suppose that $f(N_\delta(z)) \subseteq N_\eta(b)$. If $0 < |w| < \delta$ then

$$h(f(z+w)) = h(b + s(w)w) = h(b) + (h'(b) + t(s(w)w))s(w)w$$

so that

$$\frac{h(f(z+w)) - h(f(z))}{w} = (h'(b) + t(s(w)w))s(w) \to 0 \text{ as } w \to 0,$$

since $s(w) \to 0$ and $t(s(w)w) \to 0$ as $w \to 0$. □

This is essentially the same proof as in the real case. But, as we shall see (Theorem 23.1.1), the second case can only arise if f is constant on U: complex differentiation is in fact very different from real differentiation.

Corollary 20.1.5 *Suppose that g is a complex-valued function on U, which is differentiable at z. If $g(z) \neq 0$ then there is a neighbourhood $N_\delta(z) \subseteq U$ such that $g(w) \neq 0$ for $w \in N_\delta(z)$. The function $1/g$ on*

$N_\delta(z)$ is differentiable at z, with derivative $-g'(z)/g(z)^2$. Furthermore f/g is differentiable at z, with derivative

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof Since g is continuous at z, there is a neighbourhood $N_\delta(z) \subseteq U$ such that $g(w) \neq 0$ for $w \in N_\delta(z)$. Then $g(N_\delta(z)) \subseteq \mathbf{C} \setminus \{0\}$. Let $h(z) = 1/z$ for $z \in \mathbf{C} \setminus \{0\}$. Then the first result follows from the chain rule, and the second from Proposition 20.1.3. □

For example, if $p(z) = a_0 + \cdots + a^n z^n$ is a polynomial function, then p is an entire function, and $p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$. Similarly, if p and q are polynomials, and U is an open set in which q has no zeros then the rational function $r(z) = p(z)/q(z)$ is holomorphic on U, and

$$r'(z) = \frac{q(z)p'(z) - q'(z)p(z)}{q(z)^2}.$$

Exercises

20.1.1 Suppose that f is a holomorphic function on $N_1(i)$ and that $(f(z))^5 = z$ for $z \in N_1(i)$. What is $f'(i)$?

20.1.2 Suppose that f is a holomorphic function on $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Show that the set $\{n \in \mathbf{N} : f(1/(n+1)) = 1/n\}$ is finite.

20.2 The Cauchy–Riemann equations

Suppose that f is a complex-valued function on a domain U, and that $z = x + iy \in U$. We can write $f(z)$ as $u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(z)$. The functions u and v are real-valued functions of two real variables. How are differentiability properties of f related to differentiability properties of u and v?

Let us make this more explicit. Let $k : \mathbf{R}^2 \to \mathbf{C}$ be defined by setting $k((x, y)) = x + iy$; k is a linear isometry of $\mathbf{R}^2$ onto $\mathbf{C}$, considered as a real vector space. Let $j : \mathbf{C} \to \mathbf{R}^2$ be the inverse mapping. If f is a complex-valued function on U, let $\tilde{f} = j \circ f \circ k$; $\tilde{f}$ is a mapping from the open set $j(U)$ into $\mathbf{R}^2$. If $\tilde{f}(x, y) = (u(x, y), v(x, y))$, then $f(x + iy) = u(x, y) + iv(x, y)$:

$$\begin{array}{ccc} x + iy & \xrightarrow{f} & f(x + iy) = u(x, y) + iv(x, y) \\ k \uparrow & & \downarrow j \\ (x, y) & \xrightarrow{\tilde{f}} & (u(x, y), v(x, y)) \end{array}$$

Theorem 20.2.1 *Suppose that f is a complex-valued function on a domain U, and that $z_0 = x_0 + iy_0 \in U$. With the notation described above, the following are equivalent:*

(i) f is differentiable at z_0;
(ii) the function $\widetilde{f} : (x, y) \to (u(x, y), v(x, y))$ from $j(U)$ to $\mathbf{R}^2$ is differentiable at (x_0, y_0), and the partial derivatives satisfy the Cauchy–Riemann equations*:*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

If so, then

$$\frac{df}{dz}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Proof Suppose first that f is differentiable at z_0. Then

$$\begin{aligned}\frac{df}{dz}(z_0) &= \lim_{x \to 0} \frac{f(z_0 + x) - f(z_0)}{x} \\ &= \lim_{x \to 0} \frac{u(x_0 + x, y_0) - u(x_0, y_0)}{x} + i \lim_{x \to 0} \frac{v(x_0 + x, y_0) - v(x_0, y_0)}{x},\end{aligned}$$

so that the partial derivatives $(\partial u/\partial x)(x_0, y_0)$ and $(\partial v/\partial x)(x_0, y_0)$ exist, and

$$\frac{df}{dz}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

But also

$$\begin{aligned}\frac{df}{dz}(z_0) &= \lim_{y \to 0} \frac{f(z_0 + iy) - f(z_0)}{iy} \\ &= -i \lim_{y \to 0} \frac{u(x_0, y_0 + y) - u(x_0, y_0)}{y} + \lim_{y \to 0} \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0),\end{aligned}$$

so that the partial derivatives $(\partial u/\partial y)(x_0, y_0)$ and $(\partial v/\partial y)(x_0, y_0)$ exist, and

$$\frac{df}{dz}(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Thus the partial derivatives satisfy the Cauchy–Riemann equations.

Suppose that $z \in U$. Using these equations, we see that the real part of $(z - z_0)f'(z_0)$ is

$$(x - x_0)\frac{\partial u}{\partial x}(x_0, y_0) + i(y - y_0)(-i\frac{\partial u}{\partial y}(x_0, y_0))$$
$$= (x - x_0)\frac{\partial u}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial u}{\partial y}(x_0, y_0),$$

so that if we set

$$r(x, y) = u(x, y) - u(x_0, y_0) - (x - x_0)\frac{\partial u}{\partial x}(x_0, y_0) - (y - y_0)\frac{\partial u}{\partial y}(x_0, y_0)$$

then $r(x, y)$ is the real part of $f(z) - f(z_0) - (z - z_0)f'(z_0)$. Consequently, u is differentiable at (x_0, y_0). An exactly similar argument shows that the same is true for v.

Conversely, suppose that (ii) holds. Let

$$g = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Suppose that $z \in U$. Let $f(z) - f(z_0) - (z - z_0)g = h(z) + ik(z)$. Then easy calculations show that

$$h(x + iy) = u(x, y) - u(x_0, y_0) - (x - x_0)\frac{\partial u}{\partial x}(x_0, y_0) - (y - y_0)\frac{\partial u}{\partial y}(x_0, y_0),$$
$$k(x + iy) = v(x, y) - v(x_0, y_0) - (x - x_0)\frac{\partial v}{\partial x}(x_0, y_0) - (y - y_0)\frac{\partial v}{\partial y}(x_0, y_0),$$

so that

$$\frac{f(z) - f(z_0)}{z - z_0} - g = \frac{h(z) + ik(z)}{z - z_0} \to 0$$

as $z \to z_0$; hence f is differentiable at z_0, with derivative g. □

Corollary 20.2.2 *If f is holomorphic and twice continuously differentiable on U then u and v are harmonic functions; that is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Proof For

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}. \qquad \square$$

We shall see later that every holomorphic function is infinitely differentiable. Harmonic functions in Euclidean space were considered in Volume II, Section 19.8.

This result suggests a rather different approach. Suppose that $\widetilde{f}$ is differentiable at (x_0, y_0). Let $\check{f} = f \circ k$, so that $\check{f}(x, y) = f(x + iy)$. We set

$$\partial f = \frac{1}{2}\left(\frac{\partial \check{f}}{\partial x} - i\frac{\partial \check{f}}{\partial y}\right), \ \overline{\partial} f = \frac{1}{2}\left(\frac{\partial \check{f}}{\partial x} + i\frac{\partial \check{f}}{\partial y}\right).$$

Then

$$\overline{\partial} f = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right),$$

so that f is differentiable at z_0 if and only if $\overline{\partial} f(z_0) = 0$. If this is so, then

$$\partial f(z_0) = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(x_0, y_0) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)(x_0, y_0)\right) = f'(z_0).$$

We can use the Cauchy–Riemann equations and the differentiable inverse mapping theorem to prove an inverse mapping theorem for holomorphic functions. An injective holomorphic function on a domain U is said to be *univalent*: that is, it takes each value at most once on U.

Theorem 20.2.3 *Suppose that f is a univalent function on a domain U, with continuous derivative f', and suppose that $f'(z) \neq 0$ for all $z \in U$. Then $f(U)$ is an open subset of $\mathbf{C}$, the mapping $f : U \to f(U)$ is a homeomorphism, the inverse mapping $f^{-1} : f(U) \to U$ is holomorphic, and if $f(z) = w$ then $(f^{-1})'(w) = 1/f'(z)$.*

Proof Suppose that $z = x + iy \in U$. Let $r = |f'(z)|$. Since

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y),$$

it follows that

$$r^2 = \left(\frac{\partial u}{\partial x}(x,y)\right)^2 + \left(\frac{\partial v}{\partial x}(x,y)\right)^2,$$

so that there exists $0 \le \theta < 2\pi$ such that

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = r\cos\theta, \; \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) = -r\sin\theta.$$

Hence $f'(z) = r(\cos\theta + i\sin\theta)$. Thus the Jacobian $J(\widetilde{f}))$ of the mapping $\widetilde{f}$ from $j(U)$ to $j(f(U))$ is

$$\det\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r^2 > 0.$$

By the differentiable inverse mapping theorem (Volume II, Theorem 17.4.1), $j(f(U))$ is an open subset of $\mathbf{R}^2$, and the inverse mapping $\widetilde{f}^{-1} : j(f(U)) \to j(U)$ is differentiable, with derivative

$$(D\widetilde{f}^{-1})_{(u(x,y),v(x,y))} = ((D\widetilde{f})_{(x,y)})^{-1} = \begin{bmatrix} r^{-1}\cos\theta & r^{-1}\sin\theta \\ -r^{-1}\sin\theta & r^{-1}\cos\theta \end{bmatrix}.$$

Consequently, the Cauchy–Riemann equations are satisfied by f^{-1}, and f^{-1} is holomorphic; if $w = s + it = f(z) \in f(U)$ then

$$(f^{-1})'(w) = \frac{\partial \widetilde{f}^{-1}}{\partial s}(s,t) + i\frac{\partial \widetilde{f}^{-1}}{\partial t}(s,t) = \frac{\cos\theta - i\sin\theta}{r} = \frac{1}{f'(z)}. \qquad \square$$

At first sight, this looks like a strong and useful result. In fact, as we shall see, two of the hypotheses are redundant. First, the derivative of a holomorphic function on a domain is always continuous (Corollary 22.6.6), and secondly, if f is a univalent function on a domain U, then its derivative cannot take the value 0 on U (Theorem 23.6.8).

Exercises

20.2.1 Why was the chain rule not used to prove the Cauchy–Riemann equations?

20.2.2 Suppose that f is holomorphic on a domain U and that $|f|$ is constant on U. By considering $|f|^2$ and using the Cauchy–Riemann equations, show that f is constant on U.

20.2.3 Suppose that f is a non-constant holomorphic function on a domain U. Show that if $c \in \mathbf{R}$ then $\{z \in U : |f(z)| = c\}$ has an empty interior.