Cambridge University Press
978-1-107-03203-3 - A Course in Mathematical Analysis: Volume II: Metric and Topological Spaces, Functions of a Vector Variable
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Part Three

Metric and topological spaces

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11

Metric spaces and normed spaces

11.1 Metric spaces: examples

In Volume I, we established properties of real analysis, starting from the properties of the ordered field $\mathbf{R}$ of real numbers. Although the fundamental properties of $\mathbf{R}$ depend upon the order structure of $\mathbf{R}$, most of the ideas and results of the real analysis that we considered (such as the limit of a sequence, or the continuity of a function) can be expressed in terms of the distance $d(x, y) = |x - y|$ defined in Section 3.1. The concept of distance occurs in many other areas of analysis, and this is what we now investigate.

A *metric space* is a pair (X, d), where X is a set and d is a function from the product $X \times X$ to the set $\mathbf{R}^+$ of non-negative real numbers, which satisfies

1. $d(x, y) = d(y, x)$ for all $x, y \in X$ *(symmetry)*;
2. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ *(the triangle inequality)*;
3. $d(x, y) = 0$ if and only if $x = y$.

d is called a *metric*, and $d(x, y)$ is the *distance* from x to y. The conditions are very natural: the distance from x to y is the same as the distance from y to x; the distance from x to y *via* z is at least as far as any more direct route, and any two distinct points of X are a positive distance apart.

Let us give a few examples, to get us started.

Example 11.1.1 $\mathbf{R}$, with the metric $d(x, y) = |x - y|$, is a metric space, as is $\mathbf{C}$, with the metric $d(z, w) = |z - w|$.

This was established in Volume I, in Propositions 3.1.2 and 3.7.2. These metrics are called the *usual metrics*. If we consider $\mathbf{R}$ or $\mathbf{C}$ as a metric space, without specifying the metric, we assume that we are considering the usual metric.

Example 11.1.2 The Euclidean metric.

We can extend the ideas of the previous example to higher dimensions. We need an inequality.

Proposition 11.1.3 (Cauchy's inequality) *If $x, y \in \mathbf{R}^d$ then*

$$\sum_{j=1}^{d} x_j y_j \leq \left(\sum_{j=1}^{d} x_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{d} y_j^2\right)^{\frac{1}{2}}.$$

Equality holds if and only if $x_i y_j = x_j y_i$ for $1 \leq i, j \leq d$.

Proof We give the proof given by Cauchy in 1821, using Lagrange's identity:

$$\left(\sum_{j=1}^{d} x_j y_j\right)^2 + \sum_{\{(i,j):i<j\}} (x_i y_j - x_j y_i)^2 = \left(\sum_{j=1}^{d} x_j^2\right)\left(\sum_{j=1}^{d} y_j^2\right),$$

which follows by expanding the terms. This clearly establishes the inequality, and also shows that equality holds if and only if $x_i y_j = x_j y_i$ for $1 \leq i, j \leq d$. □

Corollary 11.1.4 *If $x, y \in \mathbf{R}^d$, let $d(x, y) = (\sum_{j=1}^{d}(x_j - y_j)^2)^{1/2}$. Then d is a metric on $\mathbf{R}^d$.*

Proof Conditions (i) and (iii) are clearly satisfied. We must establish the triangle inequality. First we use Cauchy's inequality to show that if $a, b \in \mathbf{R}^2$, then $d(a + b, 0) \leq d(a, 0) + d(b, 0)$:

$$\begin{aligned} d(a+b,0)^2 &= \sum_{j=1}^{d} (a_j + b_j)^2 \\ &= \sum_{j=1}^{d} a_j^2 + 2\sum_{j=1}^{d} a_j b_j + \sum_{j=1}^{d} b_j^2 \\ &\leq d(a,0)^2 + 2d(a,0).d(b,0) + d(b,0)^2 = (d(a,0) + d(b,0))^2. \end{aligned}$$

Note that it follows from the definitions that d is translation invariant: $d(a, b) = d(a + c, b + c)$. In particular, $d(a, b) = d(a - b, 0)$. If $x, y, z \in \mathbf{R}^d$, set $a = x - y$ and $b = y - z$, so that $a + b = x - z$. Then

$$d(x, z) = d(a + b, 0) \leq d(a, 0) + d(b, 0) = d(x, y) + d(y, z).$$

□

The metric d is called the *Euclidean metric*, or *standard metric*, on $\mathbf{R}^d$. When $d = 2$ or 3 it is the usual measure of distance.

We can also consider complex sequences.

Corollary 11.1.5 *If $z, w \in \mathbf{C}^d$, let $d(z,w) = (\sum_{j=1}^d |z_j - w_j|^2)^{1/2}$. Then d is a metric on $\mathbf{C}^d$.*

Proof Again, conditions (i) and (iii) are clearly satisfied, and we must establish the triangle inequality. First we show that if $z, w \in \mathbf{C}^d$ then $d(z+w,0) \le d(z,0) + d(w,0)$: using the inequality of the previous corollary,

$$d(z+w,0) = \left(\sum_{j=1}^d |z_j + w_j|^2\right)^{1/2} \le \left(\sum_{j=1}^d (|z_j| + |w_j|)^2\right)^{1/2}$$
$$\le \left(\sum_{j=1}^d |z_j|^2\right)^{1/2} + \left(\sum_{j=1}^d |w_j|^2\right)^{1/2} = d(z,0) + d(w,0).$$

Again d is translation invariant, so that $d(r,s) = d(r-s,0)$. If $r, s, t \in \mathbf{C}^d$ let $z = r - s$ and $w = s - t$, so that $z + w = r - t$ and

$$d(r,t) = d(z+w,0) \le d(z,0) + d(w,0) = d(r,s) + d(s,t).$$

□

The metric d is called the *standard metric* on $\mathbf{C}^d$.

We shall study these metrics in more detail, later.

Example 11.1.6 The discrete metric.

Let X be any set. We define $d(x,y) = 1$ if $x \ne y$ and $d(x,y) = 0$ if $x = y$. Then d is a metric on X, the *discrete* metric. If $x \in X$, there are no other points of X close to x; this means, as we shall see, that analysis on X is rather trivial.

Example 11.1.7 The subspace metric.

If (X,d) is a metric space, and Y is a subset of X, then the restriction of d to $Y \times Y$ is a metric on Y. This metric is the *subspace metric* on Y, and Y, with the subspace metric, is called a *metric subspace* of (X,d).

The subspace metric is a special case of the following. Suppose that (X,d) is a metric space and that f is an injective mapping of a set Y into X. If we set $\rho(y,y') = d(f(y), f(y'))$ then it is immediately obvious that ρ is a metric on Y. For example, we can give $(-\pi/2, \pi/2)$ the usual metric, as a subset of **R**. The mapping $j = \tan^{-1}$ is a bijection of **R** onto $(-\pi/2, \pi/2)$. Thus if we set

$$\rho(y, y') = |j(y) - j(y')| = \left|\tan^{-1}(y) - \tan^{-1}(y')\right|,$$

then ρ is a metric on **R**.

Example 11.1.8 A metric on the extended real line $\overline{\mathbf{R}}$.

We can extend the mapping j of the previous example to $\overline{\mathbf{R}}$ by setting $j(-\infty) = -\pi/2$ and $j(+\infty) = -\pi/2$. Then j is a bijection of $\overline{\mathbf{R}}$ onto $[-\pi/2, \pi/2]$, and we can again define a metric on $\overline{\mathbf{R}}$ by setting $\rho(y, y') = |j(y) - j(y')|$. Thus

$$\rho(y, \infty) = \pi/2 - \tan^{-1}(y),$$
$$\rho(-\infty, y) = \tan^{-1}(y) + \pi/2$$
$$\text{and } \rho(-\infty, \infty) = \pi.$$

Example 11.1.9 A metric on $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$.

Here is a similar construction. If $n \in \mathbf{N}$, let $f(n) = 1/n$, and let $f(+\infty) = 0$. f is an injective map of $\overline{\mathbf{N}}$ onto a closed and bounded subset of **R**. Define $\rho(x, x') = |f(x) - f(x')|$. This defines a metric on $\overline{\mathbf{N}}$:

$$\rho(m, n) = |1/m - 1/n| \text{ and } \rho(m, \infty) = 1/m.$$

Example 11.1.10 The uniform metric.

There are many cases where we define a metric on a space of functions. Here is the first and most important example. First we need some definitions. Suppose that B is a non-empty subset of a metric space (X, d). The *diameter* diam (B) of B is defined to be $\sup\{d(b, b') : b, b' \in B\}$. The set B is *bounded* if diam $(B) < \infty$. If $f : S \to (X, d)$ is a mapping and A is a non-empty subset of S, we define the *oscillation* $\Omega(f, A)$ of f on A to be the diameter of $f(A)$: $\Omega(f, A) = \sup\{d(f(a), f(a')) : a, a' \in A\}$. The function f is *bounded* if $\Omega(f, S) = \text{diam}\,(f(S)) < \infty$.

Proposition 11.1.11 *Let $B(S, X) = B_X(S)$ denote the set of all bounded mappings f from a non-empty set S to a metric space (X, d). If $f, g \in B_X(S)$, let $d_\infty(f, g) = \sup\{d(f(s), g(s)) : s \in S\}$. Then d_∞ is a metric on $B_X(S)$.*

Proof First we show that $d_\infty(f,g)$ is finite. Let $s_0 \in S$. If $s \in S$ then, by the triangle inequality,

$$\begin{aligned} d(f(s),g(s)) &\leq d(f(s),f(s_0)) + d(f(s_0),g(s_0)) + d(g(s_0),g(s)) \\ &\leq \Omega(f,S) + d(f(s_0),g(s_0)) + \Omega(g,S). \end{aligned}$$

Taking the supremum,

$$d_\infty(f,g) \leq \Omega(f,S) + d(f(s_0),g(s_0)) + \Omega(g,S) < \infty.$$

Conditions (i) and (iii) are clearly satisfied, and it remains to establish the triangle inequality. Suppose that $f,g,h \in B_X(S)$ and that $s \in S$. Then

$$d(f(s),h(s)) \leq d(f(s),g(s)) + d(g(s),h(s)) \leq d_\infty(f,g) + d_\infty(g,h).$$

Taking the supremum, $d_\infty(f,h) \leq d_\infty(f,g) + d_\infty(g,h)$. □

This metric is called the *uniform metric*.

Example 11.1.12 Pseudometrics.

We shall occasionally need to consider functions p for which the third condition in the definition of a metric is replaced by the weaker condition

(3') if $x = y$ then $p(x,y) = 0$.

In other words, we allow distinct points to be zero p-distance apart. Such a function is called a *pseudometric*. It is easy to relate a pseudometric to a metric on a quotient space.

Proposition 11.1.13 *Suppose that p is a pseudometric on a set X. The relation on X defined by setting $x \sim y$ if $d(x,y) = 0$ is an equivalence relation on X. Let q be the quotient mapping from X onto the quotient space $X/\sim$. Then there exists a metric d on $X/\sim$ such that $d(q(x),q(y)) = p(x,y)$ for $x,y \in X$.*

Proof The fact that $\sim$ is an equivalence relation on X is an immediate consequence of the symmetry property and the triangle inequality. We need a lemma, which will be useful elsewhere.

Lemma 11.1.14 *Suppose that p is a pseudometric, or a metric, on a set X, and that $a,a',b,b' \in X$. Then*

$$|p(a,b) - p(a',b')| \leq p(a,a') + p(b,b').$$

Proof Using the triangle inequality twice,

$$p(a,b) \leq p(a,a') + p(a',b) \leq p(a,a') + p(a',b') + p(b',b),$$

so that

$$p(a,b) - p(a',b') \leq p(a,a') + p(b,b').$$

Similarly

$$p(a',b') - p(a,b) \leq p(a,a') + p(b,b'),$$

which gives the result. □

We now return to the proof of the proposition. If $a \sim a'$ and $b \sim b'$ then it follows from the lemma that $p(a,b) = p(a',b')$. Thus if we define $d(q(a), q(b)) = p(a,b)$, this is well-defined. Symmetry and the triangle inequality for d now follow immediately from the corresponding properties of p. Finally if $d(q(a), q(b)) = 0$ then $p(a,b) = 0$, so that $a \sim b$ and $q(a) = q(b)$. □

We shall meet more examples of metric spaces later.

Exercises

11.1.1 If $x, y \in [0, 2\pi)$, let $d(x,y) = \min(|x-y|, 2\pi - |x-y|)$. Show that d is a metric on $[0, 2\pi)$. Define $f : [0, 2\pi) \to \mathbf{R}^2$ by setting $f(x) = (\cos x, \sin x)$. Let $\rho(f(x), f(y)) = d(x,y)$. Show that ρ is a metric on $f([0, 2\pi))$, the *arc-length metric*.

11.1.2 Suppose that p is a prime number. If r is a non-zero rational number, it can be written uniquely as $r = p^{v(r)} s/t$, where $v(r) \in \mathbf{Z}$ and s/t is a fraction in lowest terms, with neither s nor t having p as a divisor. Thus if $p = 3$ then $v(6/7) = 1$ and $v(5/18) = -2$. Let $n(r) = p^{-v(r)}$. If $r, r' \in \mathbf{Q}$, set $d_p(r,r') = n(r - r')$ for $r \neq r'$ and $d_p(r,r') = 0$ if $r = r'$. Show that d is a metric on $\mathbf{Q}$. This metric, the *p-adic metric*, is useful in number theory, but we shall not consider it further.

11.1.3 As far as I know, the next example is just a curiosity. Consider $\mathbf{R}^d$ with its usual metric d. If $x = \alpha y$ for some $\alpha \in \mathbf{R}$ (that is, x and y lie on a real straight line through the origin) set $\rho(x,y) = d(x,y)$; otherwise, set $\rho(x,y) = d(x,0) + d(0,y)$. Show that ρ is a metric on $\mathbf{R}^d$.

11.1.4 Let P_n be the power set of $\{1, \ldots, n\}$; the set of subsets of $\{1, \ldots, n\}$. Let $h(A,B) = |A \Delta B|$, where $A \Delta B$ is the symmetric difference of A and B. Show that h is a metric on P_n (the *Hamming metric*).

11.1.5 Let $P(\mathbf{N})$ be the set of subsets of $\mathbf{N}$. If A and B are distinct subsets of $\mathbf{N}$, let $d(A,B) = 2^{-j}$, where $j = \inf(A \Delta B)$, and let $d(A,A) = 0$. Show that d is a metric on $P(\mathbf{N})$ and that

$$d(A,C) \leq \max(d(A,B), d(B,C)) \text{ for } A, B, C \in P(\mathbf{N}).$$

11.1.6 Let f be the real-valued function on the extended real line $\overline{\mathbf{R}}$ defined by $f(x) = x/\sqrt{1+x^2}$ if $x \in \mathbf{R}$, $f(+\infty) = 1$ and $f(-\infty) = -1$. If $x, y \in \overline{\mathbf{R}}$, let $d(x,y) = |f(x) - f(y)|$. Show that d is a metric on $\overline{\mathbf{R}}$. Show that a sequence $(x_n)_{n=1}^\infty$ of real numbers converges to $+\infty$ as $n \to \infty$ if and only if $d(x_n, +\infty) \to 0$ as $n \to \infty$.

11.2 Normed spaces

Many of the metric spaces that we shall consider are real or complex vector spaces, and it is natural to consider metrics which relate to the algebraic structure. We shall assume knowledge of the basic algebraic properties of vector spaces and linear mappings; these are described in Appendix B. Suppose that E is a real or complex vector space. It is then natural to consider those metrics d which are *translation-invariant:* that is, $d(x+a, y+a) = d(x,y)$ for all $x, y, a \in E$. Note that this implies that

$$d(x,y) = d(x-y,0) = d(0,x-y) = d(-x,-y).$$

It is also natural to require that if we multiply by a scalar then the distance is scaled in an appropriate way: $d(\lambda x, \lambda y) = |\lambda| d(x,y)$ for all $x, y \in E$ and all scalars λ. It is easy to characterize such metrics.

A real-valued function $x \to \|x\|$ on a real or complex vector space E is a *norm* if

1. $\|x+y\| \leq \|x\| + \|y\|$ (p is *subadditive*),
2. $\|\alpha x\| = |\alpha| \|x\|$ for every scalar α and
3. $\|x\| = 0$ if and only if $x = 0$,

for α a scalar and x, y vectors in E. $(E, \|.\|)$ is then called a *normed space.* Note that $\|x\| = \|-x\|$ and that $\|0\| = \|0.0\| = 0\,\|0\| = 0$. A norm is necessarily non-negative, since $0 = \|0\| \leq \|x\| + \|-x\| = 2\,\|x\|$.

A subset C of a real or complex vector space is *convex* if whenever $x, y \in C$ and $0 \leq t \leq 1$ then $(1-t)x + ty \in C$. A real-valued function on a convex subset C of a real or complex vector space E is *convex* if whenever $x, y \in C$ and $0 \leq t \leq 1$ then

$$f((1-t)x+ty) \leq (1-t)f(x) + tf(y).$$

Proposition 11.2.1 *If $\|.\|$ is a norm on a real or complex vector space E, then $\|.\|$ is a convex function on E.*

Proof Suppose that $x, y \in E$ and $0 \leq t \leq 1$. Then

$$\|(1-t)x+ty\| \leq \|(1-t)x\| + \|ty\| = (1-t)\,\|x\| + t\,\|y\|. \qquad \square$$

Corollary 11.2.2 *The sets $U = \{x : \|x\| < 1\}$ and $B = \{x : \|x\| \leq 1\}$ are convex subsets of E.*

Theorem 11.2.3 *Suppose that d is a metric on a real or complex vector space E. Then the following are equivalent:*

(i) d is translation-invariant and satisfies $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in E$ and all scalars λ;

(ii) there exists a norm $\|.\|$ on E such that $d(x, y) = \|x - y\|$.

Proof If d is a translation-invariant metric with the desired scaling properties, and we set $\|x\| = d(x, 0)$, then $\|x\| = 0$ if and only if $x = 0$,

$$\begin{aligned} \|x + y\| &= d(x + y, 0) = d(x, -y) \leq d(x, 0) + d(0, -y) \\ &= d(x, 0) + d(y, 0) = \|x\| + \|y\|, \end{aligned}$$

and

$$\|\lambda x\| = d(\lambda x, 0) = d(\lambda x, \lambda 0) = |\lambda| d(x, 0) = |\lambda| \, \|x\| .$$

Thus (i) implies (ii).

Conversely, suppose that $\|.\|$ is a norm on E, and that we set $d(x, y) = \|x - y\|$. First we show that d is a metric on E:

$$d(x, y) = \|x - y\| = \|y - x\| = d(y, x),$$

$d(x, y) = 0$ if and only if $\|x - y\| = 0$, if and only if $x - y = 0$, if and only if $x = y$, and

$$\begin{aligned} d(x, z) = \|x - z\| &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z), \end{aligned}$$

so that the triangle inequality holds. Further,

$$d(x + a, y + a) = \|(x + a) - (y + a)\| = \|x - y\| = d(x, y)$$

so that d is translation invariant, and

$$d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = \|\lambda(x - y)\| = |\lambda| \, \|x - y\| = |\lambda| d(x, y).$$

Thus (ii) implies (i). □

A vector x in a normed space $(E, \|.\|)$ with $\|x\| = 1$ is called a *unit vector*. If y is a non-zero vector in E, then $y = \lambda y_1$, where $\lambda = \|y\|$ and y_1 is the unit vector $y / \|y\|$.