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D. J. H. Garling
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Part One

Prologue: The foundations of analysis

1

The axioms of set theory

It is probably sensible to read through this chapter fairly quickly, to find out the terminology and notation that we shall use, and then to return later to read it and think about it more carefully.

1.1 The need for axiomatic set theory

Mathematics is written in many languages, such as French, German, Russian, Chinese, and, as in the present case, English. Mathematics needs a particular precision, and within each of these languages, most of mathematics, and all the mathematics that we shall do, is written in the language of sets, using statements and arguments that are based on the grammar and logic of the predicate calculus. In this chapter we introduce the set theory that we shall use. This provides us with a framework in which to work; this framework includes a model for the natural numbers $(1, 2, 3, \ldots)$, together with tools to construct all the other number systems (rational, real and complex) and functions that are the subject of mathematical analysis.

The predicate calculus involves rules of grammar for writing 'well-formed formulae', and for providing mathematical arguments which use them. Well-formed formulae involve variables, and logical operations such as conjunction (P and Q), disjunction (P or Q (or both)), implication (P implies Q), negation (not P), and quantifiers 'there exists' and 'for all', together, in our case, with sets and the relation $\in$. We shall not describe the predicate calculus, which formalizes the everyday use of these logical operations (for example, 'P implies Q' if and only if '(not Q) implies (not P)'), but all our arguments and constructions will be based on it, and we shall give plenty of examples of well-formed formulae.[1]

[1] For a good account, see A. G. Hamilton, *Logic for Mathematicians*, Cambridge University Press, 1988.

Since the beginning of the study of set theory by Cantor in the 1870s and the introduction of Venn diagrams by Venn in 1881, the simple idea of a set has become commonplace, and young children happily manipulate sets such as {Catherine of Aragon, Ann Boleyn, Jane Seymour, Anne of Cleves, Kathryn Howard, Katherine Parr}, or more prosaically {Alice, Bob}, or the set of numbers $\{5, 13, 17, 29, 37, 41, 53, 61, 73, 89\}$. In mathematics, we consider sets of mathematical objects, such as the last of these examples. Can we not simply consider a mathematical object to be a collection of all those things which can be defined by a well-formed formula? Then a set would be something of the form 'the collection of those things a for which the well-formed formula $P(a)$ holds', where $P(x)$ is a well-formed formula with one free variable x, and conversely, each such formula would define a set. This approach is known as the *comprehension principle*. Unfortunately, it leads to contradictions. Consider the well-formed statement 'x does not belong to x'; according to the comprehension principle, there should be a set b which consists of those sets which do not belong to themselves. Does b belong to b? If it does, it fails the criterion for belonging to b, and so it does not belong to b. But if it does not belong to b, then it meets the criterion, and so it belongs to b. Thus, either way, we reach a contradiction.

This phenomenon was described by Bertrand Russell in 1901, and is known as *Russell's paradox*. It caused him a great deal of pain, as he described in his autobiography.[2] Concerning the events of May 1901, he wrote

> Cantor had a proof that there is no greatest number, and it seemed to me that the number of things in the world should be the greatest possible. Accordingly, I examined his proof with some minuteness, and endeavoured to apply it to the class of all things there are. This led me to consider those classes which are not members of themselves, and to ask whether the class of all such classes is or is not a member of itself. I found that either answer implied its contradictory.

He continued to consider the problem for several years. Describing the summers of 1903 and 1904, he wrote

> I was trying hard to solve the contradictions mentioned above. Every morning I would sit down before a blank sheet of paper. Throughout the day, with a brief interval for lunch, I would stare at the blank sheet. Often when evening came it was still empty.

Russell's paradox required a new approach to the theory of sets, which would provide a framework where Russell's paradox, and other paradoxes,

[2] *The Autobiography of Bertrand Russell*, George Allen and Unwin, 1967–69.

are avoided. In 1908, Zermelo introduced a system of axioms; these were modified in 1922 by Fraenkel and Skolem. The resulting system, known as the *Zermelo--Fraenkel axiom system* ZF, has stood the test of time, and it is the one that we shall describe and use.

1.2 The first few axioms of set theory

In Zermelo–Fraenkel set theory, the basic objects are all called *sets*, denoted by upper- or lower-case letters, and there is one relation, $\in$. Thus, if a and b are sets, then either $a \in b$, or this is not so, in which case we write $a \notin b$. (We use the symbol $/$ to mean 'not', in a similar way, for other relations.) If $a \in b$, we say that a *belongs to* b, or that a is a *member* or *element* or *point* of b, or, more simply, that a *is in* b.

The sets and the relation $\in$ are required to satisfy certain axioms, and we shall spend the rest of this chapter introducing and explaining them.

Axiom 1: The extension axiom

This states that two sets are equal if and only if they have the same elements. Thus the set with members 1, 2 and 3 and the set with members 1, 3, 2 and 1 are the same; the order in which they are listed is unimportant, as is the fact that repetition can occur. Set theory is all about membership, and about nothing else.

If a and b are sets, and every member of a is a member of b, then we say that a is a *subset* of b, or that b *contains* a, and write $a \subseteq b$ or $b \supseteq a$. Thus the extension axiom says that $a = b$ if and only if $a \subseteq b$ and $b \subseteq a$. If $a \subseteq b$ and $a \neq b$, we say that a is a *proper subset* of b, or that a is *properly contained in* b, and write $a \subset b$ or $b \supset a$.

Axiom 2: The empty set axiom

This states that there is a set with no members. The extension axiom then implies that there is only one such set: we denote it by $\emptyset$ and call it the *empty set*. It is easy to overlook the empty set: arguments involving it take on an idiosyncratic form. It also has a rather paradoxical nature, since it is a subset of every set a (if not, there is a member b of $\emptyset$ which is not in a; but $\emptyset$ has no members). Thus (looking ahead to some familiar sorts of sets) we can consider the set F of natural numbers n greater than 2 for which there exist natural numbers a, b and c with $a^n + b^n = c^n$, and we can consider the set Q of those complex quadratic polynomials of the form $z^2 + az + b$ for which the equation $z^2 + az + b = 0$ has no complex solutions. Then $F = Q$, since each is the empty set.

The next four axioms are concerned with creating new sets from old.

Axiom 3: The pairing axiom

This says that if a and b are sets then there exists a set whose members are a and b. The extension axiom again says that there is only one such set: we denote it by $\{a, b\}$. Note that $\{a, b\} = \{b, a\}$: we have an *unordered pair*. We can take $a = b$: then the set $\{a, a\}$ has only one element a. We write this set as $\{a\}$ and call it a *singleton set*.

We can use the pairing axiom to define *ordered pairs*. If a and b are sets, we define the *ordered pair* (a, b) to be the set $\{\{a\}, \{a, b\}\}$.

Proposition 1.2.1 *If (a, b) and (c, d) are ordered pairs and $(a, b) = (c, d)$, then $a = c$ and $b = d$.*

Proof The proof makes repeated use of the extension axiom. First, suppose that $a = b$. Then $(a, b) = \{\{a\}\} = \{\{c\}, \{c, d\}\}$, and so $\{c, d\} = \{a\}$, and $a = c = d$. Thus $a = b = c = d$. Similarly, if $c = d$ then $a = b = c = d$.

Finally, suppose that $a \neq b$ and $c \neq d$. Since $\{a\} \in (c, d)$, either $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. But if $\{a\} = \{c, d\}$ then $c = a = d$, giving a contradiction. Thus $\{a\} = \{c\}$ and $a = c$. Since $\{a, b\} \in (c, d)$, either $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. But if $\{a, b\} = \{c\}$, then $a = c = b$, giving a contradiction. Thus $\{a, b\} = \{c, d\}$, and so $b = c$ or $b = d$. But if $b = c$ then $b = c = a$, giving a contradiction. Thus $b = d$. □

If A is a set, then all its members are sets, and they, in turn, can have members.

Axiom 4: The union axiom

This says that there is a set whose elements are exactly the sets which are members of members of A. We denote this set by $\cup_{a \in A} a$ (here a is a variable, so we could as well write $\cup_{x \in A} x$) and call it the *union* of the members of A. The essential feature of this axiom is that the sets whose members make up the union must all be members of a single set; we cannot form the union of *all* sets since, as we shall see, there is no set to which all sets belong. If A and B are sets, we can consider the set $\cup_{C \in \{A, B\}} C$. This is the set whose elements are either in A or in B: we write this as $A \cup B$.

Axiom 5: The power set axiom

There is an essential difference between the statements $b \in A$ (b is a member of A) and $b \subseteq A$ (b is a subset of A). The power set axiom states that if A is a set, then there exists a set, the *power set* $P(A)$ of A, whose elements are the subsets of A. Thus $b \in P(A)$ if and only if $b \subseteq A$. For example, the elements of $P(\{a, b\})$ are $\emptyset$, $\{a\}$, $\{b\}$ and $\{a, b\}$, and the ordered pair $(a, b) = \{\{a\}, \{a, b\}\}$ is an element of $P(P(\{a, b\}))$.

Axiom 6: The separation axiom

This is particularly important, and is an axiom that is used all the time in mathematics. It states that if A is a set and $Q(x)$ is a well-formed formula, then there exists a subset of A whose elements are just those members a of A for which $Q(a)$ holds. By extensionality, there is only one such set; we denote it by $\{x \in A : Q(x)\}$. With this axiom in place, we can use the argument of Russell's paradox to show that there is no universal set to which every set belongs.

Theorem 1.2.2 *There is no set Ω such that if a is a set then $a \in \Omega$.*

Proof Suppose that such a set were to exist. Then the formula $x \notin x$ is a well-formed formula, and so there exists a set $b = \{x \in \Omega : x \notin x\}$. Does $b \in b$? If it does, it fails the criterion for membership, giving a contradiction. If it does not, then it meets the criterion, and so belongs to b, giving another contradiction. This exhausts all possibilities, and so no such universal set can exist. □

Let us give some more examples of the use of the separation axiom. Suppose that A and B are sets. The expression $x \in B$ is a well-formed formula, and so the set $\{x \in A : x \in B\}$ is a subset of A, the *intersection* of A and B, denoted by $A \cap B$. Note that $A \cap B = B \cap A = \{x \in B : x \in A\}$, since a set c is an element of either intersection if and only if it belongs to both A and B. We say that A and B are *disjoint* if $A \cap B = \emptyset$; A and B are disjoint if A and B have no member in common. Similarly, the expression $x \notin B$ is a well-formed formula, and so the set $\{x \in A : x \notin B\}$ is a subset of A, the *set difference* $A \setminus B$. $A \setminus B$ is also called the *relative complement* of B in A. It frequently happens that we consider a particular set A, say, and are only concerned with subsets of A. In this case, if $B \subseteq A$, then we denote $A \setminus B$ by $C(B)$, or B^c, and call it the *complement* of B.

We can extend the notion of intersection considerably. Suppose that A is a set. The expression 'for all $a \in A$, $x \in a$' is a well-formed formula with a a bound variable and x a free variable, and so we can form the set

$$\{x \in \cup_{a \in A} a : \text{for all } a \in A, x \in a\}.$$

This is the *intersection* $\cap_{a\in A}a$ of all the sets a that belong to A: $b \in \cap_{a\in A}a$ if and only if $b \in a$, for each $a \in A$. Here again a is a variable, and we could also write $\cap_{x\in A}x$. We must reconcile the two definitions of intersection that we have made: this is easy because $A \cap B = \cap_{x\in\{A,B\}}x$.

A word about notation here. Our aim will be to be accurate and clear without being pedantic. Suppose that A is a set. For each $a \in A$, we can form the intersection $\cap_{\alpha\in a}\alpha$. Using the separation axiom, we can then define the set I whose elements are exactly these intersections, and can then form the set $\cup_{i\in I}i$. In fact, we write this in the form

$$\cup_{a\in A}(\cap_{\alpha\in a}\alpha),$$

and use other similar expressions. In the same way, we shall use natural variations of the notation $\{x \in A : Q(x)\}$ to denote sets whose existence is ensured by the separation axiom; but in each case such a set is a subset of a given set, and it can be written, at greater length, in the form $\{x\in A : Q(x)\}$.

From now on, we shall define sets without appealing to the axioms to ensure that they are in fact sets. It is a useful exercise for the reader to consider, in each case, how suitable justification can be given.

It is unfortunately the case that the separation axiom is not strong enough for all purposes, and another axiom, the *replacement axiom*, is needed. We shall defer discussion of this and of the other axioms of ZF, until later. Let us first see what we can do with the axioms that we now have.

Exercises

Suppose that A, B, C, D are sets.

1.2.1 Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
1.2.2 Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
1.2.3 Show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
1.2.4 Which of the following statements are necessarily true?
 (a) $P(A \cap B) = P(A) \cap P(B)$.
 (b) $P(A \cup B) = P(A) \cup P(B)$.
1.2.5 Define a set I such that $\cup_{i\in I}i = \cup_{a\in A}(\cap_{\alpha\in a}\alpha)$.
1.2.6 Does $\cup_{a\in A}(\cap_{\alpha\in a}\alpha)$ necessarily contain $\cap_{a\in A}(\cup_{\alpha\in a}\alpha)$? Is $\cup_{a\in A}(\cap_{\alpha\in a}\alpha)$ necessarily contained in $\cap_{a\in A}(\cup_{\alpha\in a}\alpha)$?
1.2.7 The *symmetric difference* $a\Delta b$ of two sets a and b is the set $(a\setminus b)\cup(b\setminus a)$. Establish the following:
 (a) $A\Delta B = (A \cup B) \setminus (A \cap B)$.

(b) $A\Delta B = B\Delta A$.
(c) $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.
(d) $A\Delta\emptyset = A$.
(e) $A\Delta A = \emptyset$.

1.3 Relations and partial orders

The *Cartesian product* $A \times B$ of two sets A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. More formally,

$$A \times B = \{x \in P(P(A \cup B)) : \text{there exists } a \in A \text{ and there exists } b \in B \text{ such that } x = \{\{a\}, \{a, b\}\}\}.$$

(The term *Cartesian* honours René Descartes, who introduced coordinates to the plane, so that points in the plane are represented by ordered pairs of real numbers; the plane is thus represented as the Cartesian product of two copies of the set of real numbers.)

A *relation* on $A \times B$ is then simply a subset R of $A \times B$. It is customary to write aRb if $(a, b) \in R$. The set

$$\{a \in A : \text{ there exists } b \in B \text{ such that } (a, b) \in R\}$$

is then called the *domain* of R, and the set

$$\{b \in B : \text{ there exists } a \in A \text{ such that } (a, b) \in R\}$$

is called the *range* of R. A relation on $A \times A$ is called a *relation on* A.

Let us give some examples. First, if A is a set then

$$\in_A = \{(b, B) \in A \times P(A) : b \in B\}$$

is a relation on $A \times P(A)$. Recall that we introduced the relation $\in$ on the collection of all sets, which we have seen is not a set; ϵ_A is the restriction to a set and its subsets.

Secondly, if A is a set then

$$\subseteq_A = \{(B, C) \in P(A) \times P(A) : B \subseteq C\}$$

is a relation on $P(A)$. This is an example of a partial order relation. An order $\leq$ on a set A is a *partial order* or *partial order relation* if

(i) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity), and
(ii) $a \leq b$ and $b \leq a$ if and only if $a = b$.

If $a \leq b$ then we say that a is *less than or equal* to b, or that b is *greater than or equal* to a, and we also write $b \geq a$.

Partial order relations play an important part in analysis. We make some definitions concerning partial orders here, and will consider them in more detail later.

Suppose that $\leq$ is a partial order on a set A, that $a \in A$ and that B is a subset of A.

- a is an *upper bound* of B if $b \leq a$ for all $b \in B$.
- a is a *lower bound* of B if $a \leq b$ for all $b \in B$.

An upper bound of B need not belong to B. If it does, it is the *greatest* element of B. B has at most one greatest element, but may have no greatest element. *Least* elements are defined in the same way.

- a ia a *maximal* element of B if $a \in B$, and if $b \in B$ and $a \leq b$ then $a = b$.
- a ia a *minimal* element of B if $a \in B$, and if $b \in B$ and $b \leq a$ then $a = b$.

A greatest element of B is a maximal element of B, but the converse need not hold.

- a is the *supremum*, or *least upper bound*, of B if a is an upper bound of B, and if c is an upper bound of B, then $a \leq c$. In other words, a is the least element of the set of upper bounds of B.
- a is the *infimum*, or *greatest lower bound*, of B if a is a lower bound of B, and if c is an lower bound of B, then $c \leq a$. In other words, a is the greatest element of the set of lower bounds of B.

B has at most one least upper bound, but may have no least upper bound. If a is the least upper bound of B then a may or may not be an element of B. If a is an element of B, then a is the least upper bound of B if and only if a is the greatest element of B.

If $a \leq b$ or $b \leq a$ then we say that a and b are *comparable*. In general, not all pairs are comparable. If, however, any two elements of A are comparable, then we say that the relation is a *total order*. As an example, the usual order on the set of natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$ (which we shall consider in Section 2.1) is a total order.

The definition of the notion of partial order includes equality. There is a closely related notion which forbids equality. Suppose that $\leq$ is a partial order relation on a set A. Then the relation

$$\{(a, b) \in A \times A : a \leq b \text{ and } a \neq b\}$$