1.1 Introduction

This chapter considers elements of matrix algebra, knowledge of which is essential for discussions throughout this book. This body of mathematics centres around the concepts of Kronecker products and vecs of a matrix. From the elements of an \( m \times n \) matrix \( A = \{a_{ij}\} \) and a \( p \times q \) matrix \( B = \{b_{ij}\} \), the Kronecker product forms a new \( mp \times nq \) matrix. The vec operator forms a column vector from the elements of a given matrix by stacking its columns one underneath the other. This chapter discusses several new operators that are derived from these basic operators.

The operator, which I call the cross-product operator, takes the sum of Kronecker products formed from submatrices of two given matrices. The rvec operator forms a row vector by stacking the rows of a given matrix alongside each other. The generalized vec operator forms a new matrix from a given matrix by stacking a certain number of its columns, taken as a block, under each other. The generalized rvec operator forms a new matrix by stacking a certain number of rows, again taken as a block, alongside each other.

Although it is well known that Kronecker products and vecs are intimately connected, this connection also holds for rvec and generalised operators as well. The cross-product operator, as far as I know, is being introduced by this book. As such, I present several theorems designed to investigate the properties of this operator. This book’s approach is to list, without proof, well-known properties of the mathematical operator or concept in hand. However, I give a proof whenever I present the properties of a new operator or concept, a property in a different light, or something new about a concept.
1.2 Kronecker Products

Let $A = \{a_{ij}\}$ be an $m \times n$ matrix and $B$ be a $p \times q$ matrix. The $mp \times nq$ matrix given by

$$
\begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
$$

is called the **Kronecker product** of $A$ and $B$, denoted by $A \otimes B$.

Well-known properties of Kronecker products are as follows:

\[ A \otimes (B \otimes C) = (A \otimes B) \otimes C = A \otimes B \otimes C; \]
\[ (A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D; \text{ and} \]
\[ (A \otimes B)(C \otimes D) = AC \otimes BD \text{ provided } AC \text{ and } BD \text{ exist.} \quad (1.1) \]

The transpose of a Kronecker product is the Kronecker product of transposes

\[ (A \otimes B)' = A' \otimes B'. \]

If $A$ and $B$ are non-singular, the inverse of a Kronecker product is the Kronecker product of the inverses

\[ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \]

If $A$ is an $n \times n$ matrix and $B$ is an $p \times p$ matrix, then

\[ \text{tr}(A \otimes B) = \text{tr}A \cdot \text{tr}B, \]

where the determinant of the Kronecker product is given by

\[ |A \otimes B| = |A|^p \cdot |B|^n. \]

Notice that generally, this operator does not obey the commutative law. That is, $A \otimes B \neq B \otimes A$. One exception to this rule is if $a$ and $b$ are column vectors, not necessarily of the same order, then

\[ a' \otimes b = b \otimes a' = ba'. \quad (1.2) \]

This exception allows us to write $A \otimes b$ in an interesting way, where $A$ is an $m \times n$ matrix and $b$ is a column vector. Partition $A$ into its rows so

\[ A = \begin{bmatrix} a_1' & \cdots & a_m' \\
\end{bmatrix} \]
where the notation we use for the $i$th row of a matrix throughout this book is $a^i$. Thus, from our definition of a Kronecker product

$$A \otimes b = \begin{pmatrix} a^1 \otimes b \\ \vdots \\ a^m \otimes b \end{pmatrix}.$$ 

By using Equation 1.2, we can write

$$A \otimes b = \begin{pmatrix} b \otimes a^1 \\ \vdots \\ b \otimes a^m \end{pmatrix}.$$ 

As far as partitioned matrices are concerned, suppose we partition $A$ into submatrices as follows:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1K} \\ \vdots & \ddots & \vdots \\ A_{L1} & \cdots & A_{LK} \end{pmatrix}.$$ 

Therefore, from our definition it is clear that

$$A \otimes B = \begin{pmatrix} A_{11} \otimes B & \cdots & A_{1K} \otimes B \\ \vdots & \ddots & \vdots \\ A_{L1} \otimes B & \cdots & A_{LK} \otimes B \end{pmatrix}. \quad (1.3)$$

Likewise, suppose we partition $B$ into an arbitrary number of submatrices, say,

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & \ddots & \vdots \\ B_{s1} & \cdots & B_{sr} \end{pmatrix}.$$ 

Then, in general,

$$A \otimes B \neq \begin{pmatrix} A \otimes B_{11} & \cdots & A \otimes B_{1r} \\ \vdots & \ddots & \vdots \\ A \otimes B_{s1} & \cdots & A \otimes B_{sr} \end{pmatrix}.$$ 

One exception to this rule can be formulated as follows: Suppose $B$ is a $p \times q$ matrix and we write $B = (B_1 \ldots B_j)$, where each submatrix of $B$ has $p$ rows.
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Furthermore, let $a$ be any column vector, say $m \times 1$. Then,

$$a \otimes B = \begin{pmatrix} a_1(B_1 \ldots B_r) \\ \vdots \\ a_m(B_1 \ldots B_r) \end{pmatrix} = \begin{pmatrix} a_1B_1 & \ldots & a_1B_r \\ \vdots & \ddots & \vdots \\ a_mB_1 & \ldots & a_mB_r \end{pmatrix} = (a \otimes B_1 \ldots a \otimes B_r).$$ (1.4)

Staying with the same partitioning of $B$, consider $A$ a $m \times n$ matrix partitioned into its columns $A = (a_1 \ldots a_n)$. Therefore, using Equations 1.3 and 1.4, it is clear that

$$A \otimes B = (a_1 \otimes B_1 \ldots a_n \otimes B_1 \ldots a_1 \otimes B_r \ldots a_n \otimes B_r).$$

If, for example, $B$ is partitioned into its columns, then $B = (b_1 \ldots b_q)$, so we can write

$$A \otimes B = (a_1 \otimes b_1 \ldots a_1 \otimes b_q \ldots a_n \otimes b_1 \ldots a_n \otimes b_q).$$ (1.5)

Another exception to the rule is $a' \otimes B$, where we now partition $B$ as $B = (B'_1 \ldots B'_s)'$ and each submatrix has $q$ columns. Therefore,

$$a' \otimes B = \begin{pmatrix} a' \otimes B_1 \\ \vdots \\ a' \otimes B_s \end{pmatrix}.$$

If $A$ is $m \times n$, then

$$A \otimes B = \begin{pmatrix} a' \otimes B_1 \\ \vdots \\ a' \otimes B_s \\ a'' \otimes B_1 \\ \vdots \\ a'' \otimes B_s \\ \vdots \\ a''' \otimes B_1 \\ \vdots \\ a''' \otimes B_s \end{pmatrix}.$$
where, as before, $a^i$ refers to the $i$th row of $A$, $i = 1, \ldots, m$. If $B$ is partitioned into its rows, then

$$A \otimes B = \begin{pmatrix} a^1 \otimes b^1 \\ \vdots \\ a^m \otimes b^1 \\ \vdots \\ a^1 \otimes b^p \\ \vdots \\ a^m \otimes b^p \end{pmatrix}$$ (1.6)

where $b^j$ refers to this $j$th row of $B$, $j = 1, \ldots, p$.

Let $x$ be a column vector and $A$ a matrix. As a consequence of these results, the $i$th row of $x^t \otimes A$ is $x^t \otimes a^i$, where $a^i$ is the $i$th row of $A$, and the $j$th column of $x \otimes A$ is $x \otimes a_j$, where $a_j$ is the $j$th column of $A$.

Another useful property for Kronecker products is this: Suppose $A$ and $B$ are $m \times n$ and $p \times q$ matrices respectively, and $x$ is any column vector. Then,

$$A(I_n \otimes x^t) = (A \otimes 1)(I_n \otimes x^t) = A \otimes x^t$$

$$(x \otimes I_p)B = (x \otimes I_p)(1 \otimes B) = x \otimes B,$$

where $I_n$ is the $n \times n$ identity matrix.

We can use these results to prove that for $a$, a $n \times 1$ column vector and $b$ a $p \times 1$ column vector,

$$(a^t \otimes I_G)(b^t \otimes I_{nG}) = b^t \otimes a^t \otimes I_G.$$

Clearly,

$$(a^t \otimes I_G)(b^t \otimes I_{nG}) = (a^t \otimes I_G)(b^t \otimes I_n \otimes I_G) = a^t (b^t \otimes I_n) \otimes I_G = (1 \otimes a^t)(b^t \otimes I_n) \otimes I_G = b^t \otimes a^t \otimes I_G.$$

Another notation used throughout this book is: I represent the $i$th column of the $n \times n$ identity matrix $I_n$ by $e_i^n$ and the $j$th row of this identity matrix by $e_i^j$. Using this notation, a result that we find useful in our future work is given by our first theorem.

**Theorem 1.1** Consider the $n \times m$ matrix given by

$$
\begin{pmatrix}
O & e_i^n \\
O & e_i^{n \times (m-p)}
\end{pmatrix}
$$
for \( i = 1, \ldots, n \). Then,
\[
I_n \otimes e^m_m = \left( O e^n_1 \ O \ldots \ O e^n_n \ O \right).
\]

**Proof:** We have
\[
I_n \otimes e^m_m = \left( e^n_1 \otimes e^m_p \ldots e^n_n \otimes e^m_p \right) = \left( e^m_p \otimes e^n_1 \ldots e^m_p \otimes e^n_n \right)
= \left( O e^n_1 \ O \ldots \ O e^n_n \ O \right).
\]

\[\square\]

### 1.3 Cross-Product of Matrices

Much of this book’s discussions involve partitioned matrices. A matrix operator that I find very useful when working with such matrices is the cross-product operator. This section introduces this operator and presents several theorems designed to portray its properties.

Let \( A \) be an \( mG \times p \) matrix and \( B \) be an \( nG \times q \) matrix. Partition these matrices as follows:

\[
A = \begin{pmatrix}
A_1 \\
\vdots \\
A_G
\end{pmatrix}, \quad B = \begin{pmatrix}
B_1 \\
\vdots \\
B_G
\end{pmatrix}
\]

where each submatrix \( A_i \) of \( A \) is \( m \times p \) for \( i = 1, \ldots, G \) and each submatrix \( B_j \) of \( B \) is \( n \times q \) for \( j = 1, \ldots, G \). The **cross-product** of \( A \) and \( B \) denoted by \( A \tau_{Gmn} B \) is the \( mn \times pq \) matrix given by

\[
A \tau_{Gmn} B = A_1 \otimes B_1 + \cdots + A_G \otimes B_G.
\]

Notice the first subscript attached to the operator refers to the number of submatrices in the partitions of the two matrices, the second subscript refers to the number of rows in each submatrix of \( A \), and the third subscript refers to the number of rows in each of the submatrices of \( B \).

A similar operator can be defined when the two matrices in question are partitioned into a row of submatrices, instead of a column of submatrices as previously discussed. Let \( C \) be a \( p \times mG \) matrix and \( D \) be a \( q \times nG \) matrix, and partition these matrices as follows:

\[
C = (C_1 \ldots C_G) \quad D = (D_1 \ldots D_G),
\]

where each submatrix \( C_i \) of \( C \) is \( p \times m \) for \( i = 1, \ldots, G \) and each submatrix \( D_j \) of \( D \) is \( q \times n \) for \( j = 1, \ldots, G \). Then, the column cross-product is defined as

\[
C \tau_{Gmn} D = C_1 \otimes D_1 + \cdots + C_G \otimes D_G.
\]
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The operator $\tau$ is the relevant operator to use when matrices are partitioned into a ‘column’ of submatrices, whereas $\overline{\tau}$ is the appropriate operator to use when matrices are partitioned into a ‘row’ of submatrices. The two operators are intimately connected as

$$(A \tau_{Gmn} B)' = A'_1 \otimes B'_1 + \cdots + A'_G \otimes B'_G = A' \tau_{Gmn} B'.$$

In this book, theorems are proved for $\tau$ operator and the equivalent results for the $\overline{\tau}$ operator can be obtained by taking transposes.

Sometimes, we have occasion to take the cross-products of very large matrices. For example, suppose $A$ is $mrG \times p$ and $B$ is $nG \times q$ as previously shown. Thus, if we partition $A$ as

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_G \end{pmatrix},$$

each of the submatrices in this partition is $mr \times p$. To avoid confusion, signify the cross-product between $A$ and $B$, namely $A_1 \otimes B_1 + \cdots + A_G \otimes B_G$ as $A \tau_{G,mr,n} B$, and the cross-product between $B$ and $A$, $B_1 \otimes A_1 + \cdots + B_G \otimes A_G$ as $B \tau_{G,n,m} A$.

Notice that in dealing with two matrices $A$ and $B$, where $A$ is $mG \times p$ and $B$ is $mG \times q$, then it is possible to take two cross-products $A \tau_{Gmn} B$ or $A \tau_{mGn} B$, but, of course, these are not the same. However, the following theorem shows that in some cases the two cross-products are related.

**Theorem 1.2** Let $A$ be a $mG \times p$ matrix, $B$ be an $ns \times q$ matrix, and $D$ be a $G \times s$ matrix. Then,

$$B \tau_{snm}(D' \otimes I_m) A = (D \otimes I_n) B \tau_{Gnm} A.$$

**Proof:** Write

$$D = (d_1 \ldots d_s) = \begin{pmatrix} d^1 \\ \vdots \\ d^G \end{pmatrix}.$$

Then,

$$(D \otimes I_n) B = \begin{pmatrix} d^1 \otimes I_n \\ \vdots \\ d^G \otimes I_n \end{pmatrix} B = \begin{pmatrix} (d^1 \otimes I_n) B \\ \vdots \\ (d^G \otimes I_n) B \end{pmatrix}.$$
Partition $A$ as

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_G \end{pmatrix}$$

where each submatrix $A_i$ is $m \times p$. Then,

$$(D \otimes I_n) B \tau_{Gnm} A = (d^1 \otimes I_n) B \otimes A_1 + \cdots + (d^G \otimes I_n) B \otimes A_G.$$ 

Now

$$(D' \otimes I_m) A = \begin{pmatrix} d'_1 \otimes I_m \\ \vdots \\ d'_s \otimes I_m \end{pmatrix} A = \begin{pmatrix} (d'_1 \otimes I_m) A \\ \vdots \\ (d'_s \otimes I_m) A \end{pmatrix}.$$ 

But,

$$(d'_j \otimes I_m) A = (d'_j I_m \cdots d'_G I_m) \begin{pmatrix} A_1 \\ \vdots \\ A_G \end{pmatrix} = d_j A_1 + \cdots + d_G A_G,$$

so when we partition $B$ as

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_s \end{pmatrix}$$

where each submatrix $B_i$ is $n \times q$, we have

$$B \tau_{nm} (D' \otimes I_m) A$$

$$= B_1 \otimes (d_{11} A_1 + \cdots + d_{1G} A_G) + \cdots + B_s \otimes (d_{s1} A_1 + \cdots + d_{sG} A_G)$$

$$= B_1 \otimes d_{11} A_1 + \cdots + B_s \otimes d_{s1} A_1 + B_1 \otimes d_{1G} A_G + \cdots + B_s \otimes d_{sG} A_G$$

$$= (d_{11} B_1 + \cdots + d_{1G} B_s) \otimes A_1 + \cdots + (d_{G1} B_1 + \cdots + d_{Gs} B_s) \otimes A_G$$

$$= (d' \otimes I_n) B \otimes A_1 + \cdots + (d'^G \otimes I_n) B \otimes A_G.$$ 

In the following theorems, unless specified, $A$ is $mG \times p$ and $B$ is $nG \times q$, and

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_G \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_G \end{pmatrix}$$

(1.7)
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where each submatrix $A_i$ of $A$ is $m \times p$ and each submatrix $B_j$ of $B$ is $n \times q$, for $i = 1, \ldots, G$ and $j = 1, \ldots, G$. The proofs of these theorems are derived using the properties of Kronecker products.

**Theorem 1.3** Partition $A$ differently as

$$A = (C \ D \ldots \ F)$$

where each submatrix $C, D, \ldots, F$ has $mG$ rows. Then,

$$A \tau_{Gmn} B = (C \tau_{Gmn} B \ D \tau_{Gmn} B \ldots F \tau_{Gmn} B).$$

**Proof:** From our definition,

$$A \tau_{Gmn} B = A_1 \otimes B_1 + \cdots + A_G \otimes B_G.$$  

Writing $A_i = (C_i \ D_i \ldots F_i)$ for $i = 1, \ldots, G$, we have from the properties of Kronecker products that

$$A_i \otimes B_i = (C_i \otimes B_i \ D_i \otimes B_i \ldots F_i \otimes B_i).$$

The result follows immediately. ■

**Theorem 1.4** Let $A$ and $B$ be $mG \times p$ matrices, and let $C$ and $D$ be $nG \times q$ matrices. Then,

$$(A + B) \tau_{Gmn} C = A \tau_{Gmn} C + B \tau_{Gmn} C$$

and

$$A \tau_{Gmn} (C + D) = A \tau_{Gmn} C + A \tau_{Gmn} D.$$  

**Proof:** Clearly,

$$(A + B) \tau_{Gmn} C = (A_1 + B_1) \otimes C_1 + \cdots + (A_G + B_G) \otimes C_G$$

$$= A_1 \otimes C_1 + \cdots + A_G \otimes C_G + B_1 \otimes C_1 + \cdots + B_G \otimes C_G$$

$$= A \tau_{Gmn} C + B \tau_{Gmn} C.$$  

The second result is proved similarly. ■

**Theorem 1.5** Let $A$ and $B$ be specified in Equation 1.7, let $C, D, E, F$ be $p \times r, q \times s, r \times m$, and $s \times n$ matrices, respectively. Then,

$$(A \tau_{Gmn} B)(C \otimes D) = AC \tau_{Gmn} BD$$
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and

\[(E \otimes F)(A_{\tau Gmn}B) = (I_G \otimes E)A_{\tau Grs}(I_G \otimes F)B.\]

**Proof:** Clearly,

\[(A_{\tau Gmn}B)(C \otimes D) = (A_1 \otimes B_1 + \cdots + A_G \otimes B_G)(C \otimes D)
= A_1 C \otimes B_1 D + \cdots + A_G C \otimes B_G D
= \begin{pmatrix} A_1 C \\ \vdots \\ A_G C \end{pmatrix} \tau_{Gmn} \begin{pmatrix} B_1 D \\ \vdots \\ B_G D \end{pmatrix} = AC \tau_{Gmn} B D.
\]

Likewise,

\[(E \otimes F)(A_{\tau Gmn}B) = (E \otimes F)(A_1 \otimes B_1 + \cdots + A_G \otimes B_G)
= EA_1 \otimes FB_1 + \cdots + EA_G \otimes FB_G
= \begin{pmatrix} EA_1 \\ \vdots \\ EA_G \end{pmatrix} \tau_{Grs} \begin{pmatrix} FB_1 \\ \vdots \\ FB_G \end{pmatrix}
= (I_G \otimes E)A \tau_{Grs} (I_G \otimes F)B.\]

A standard notation that is regularly used in this book is

\[A_i = \text{i-th row of the matrix } A\]
\[A_j = \text{j-th column of the matrix } A.\]

For the next theorem, it is advantageous to introduce a new notation that we will find useful for our work throughout most chapters. We are considering \(A\) a \(mG \times p\) matrix, which we have partitioned as

\[A = \begin{pmatrix} A_1 \\ \vdots \\ A_G \end{pmatrix} \]

where each submatrix \(A_j\) in this partitioning is \(m \times p\). Thus, we denoted by \(A^{(j)}\) the \(G \times p\) matrix given by

\[A^{(j)} = \begin{pmatrix} (A_1)_j \\ \vdots \\ (A_G)_j \end{pmatrix}. \tag{1.8}\]