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Ladislav Šamaj and Zoltán Bajnok
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INTRODUCTION TO THE STATISTICAL PHYSICS OF INTEGRABLE MANY-BODY SYSTEMS

Including topics not traditionally covered in the literature, such as (1 + 1)-dimensional quantum field theory and classical two-dimensional Coulomb gases, this book considers a wide range of models and demonstrates a number of situations to which they can be applied.

Beginning with a treatise on non-relativistic one-dimensional continuum Fermi and Bose quantum gases of identical spinless particles, the book describes the quantum inverse-scattering method and the analysis of the related Yang–Baxter equation and integrable quantum Heisenberg models. It also discusses systems within condensed matter physics, the complete solution of the sine–Gordon model and modern trends in the thermodynamic Bethe ansatz.

Each chapter concludes with problems and solutions to help consolidate the reader's understanding of the theory and its applications. Basic knowledge of quantum mechanics and equilibrium statistical physics is assumed, making this book suitable for graduate students and researchers in statistical physics, quantum mechanics and mathematical and theoretical physics.

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This book forms part of an informal series of books, all of which originated as review articles published in *Acta Physica Slovaca*. The journal can be freely accessed at www.physics.sk/aps.

Vladimir Bužek, editor of the journal

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Preface

In classical mechanics, a dynamical system of interacting bodies with $2N$-dimensional phase space is said to be integrable if there exist N conserved functions (charges) whose Poisson brackets vanish. For an integrable system in quantum field theory (QFT) there exists an infinite set of commuting conserved charges. The existence of the conserved charges allows us to solve the physical system exactly and in this way to describe the modeled phenomena without any approximation. Although the integrability is restricted to low dimensions, the exact solution often provides general information about the physical phenomena. At present, we know precisely how to generate systematically integrable models and how to solve them, explicitly or implicitly in the form of integral equations.

Integrable models cover many domains of quantum mechanics and statistical physics:

- Non-relativistic one-dimensional (1D) continuum Fermi and Bose quantum gases with specific types of singular and short-range interactions.
- 1D lattice and continuum quantum models of condensed-matter physics, like the Heisenberg model of interacting quantum spins, the Hubbard model of hopping electrons with one-site interactions between electrons of opposite spins, the Kondo model describing the interaction of a conduction band with a localized spin impurity, microscopic models of superconductors, etc.
- Relativistic models of QFT in a (1+1)-dimensional spacetime like the sine–Gordon model and its fermionic analog, the Thirring model, and so on.
- Two-dimensional (2D) lattice and continuum classical models in thermal equilibrium like the lattice Ising model of interacting nearest-neighbor ± 1 spins, the six- and eight-vertex models, the continuum Coulomb gas of ± 1 charges interacting by a logarithmic potential, etc.

The solution of the equilibrium statistical mechanics of an integrable classical model formulated on a 2D lattice consists of the diagonalization of a row-to-row transfer matrix whose largest eigenvalue determines the thermodynamic limit of the free energy. From this point of view, the problem resembles technically that of finding the energy spectrum of a quantum-mechanical model in spatial dimension reduced by one.

Integrable systems can be either homogeneous, i.e. formulated in a finite domain with periodic boundary conditions or taken as infinite (the thermodynamic limit), or inhomogeneous, e.g. in the presence of a hard-wall boundary impenetrable to particles. In this book, we restrict ourselves to homogeneous systems.

The complete solution of an integrable 1D quantum-mechanical model proceeds in several steps:

- As a first step, one reduces the problem of calculating the spectrum of a Hamiltonian to solving a set of coupled algebraic equations. In this way the original problem of exponential complexity is transformed to one of polynomial complexity. The coupled equations are known, for historical reasons, as the Bethe ansatz equations and have an adjective which depends on the type of the system under consideration or on the applied method. The adjectives are "coordinate" for spinless particles treated in the direct space format, "nested" for particles with internal degrees of freedom like spin, "algebraic" for an inverse-scattering formulation, etc. and they can be combined, too.
- The next step is to find the solution of the Bethe equations which corresponds to the ground state, i.e. the eigenstate of the Hamiltonian with the lowest energy, and the zero-temperature thermodynamics. A substantial simplification arises in the thermodynamic limit within a continuum procedure.
- The third step consists of the construction of low-lying excitations upon the ground state and in finding the asymptotic expression for their energy in the thermodynamic limit.
- The fourth step is the derivation of the thermodynamics (the free energy) for the system at temperature $T > 0$ (the "thermodynamic Bethe ansatz").
- The final step is the evaluation of correlation functions of interacting bodies at arbitrary distance. This topic goes beyond the scope of the present book.

In the following paragraphs, we shall briefly summarize some milestones in the history of the statistical physics of integrable many-body systems.

The most important integrable system was certainly the quantum-mechanical model of magnetism proposed by Heisenberg [1]. The Heisenberg Hamiltonian of N interacting particles with spin $\frac{1}{2}$ on a 1D chain reads

$$H = -\frac{1}{2}\sum_{n=1}^{N}\left(J_x\boldsymbol{\sigma}_n^x\boldsymbol{\sigma}_{n+1}^x + J_y\boldsymbol{\sigma}_n^y\boldsymbol{\sigma}_{n+1}^y + J_z\boldsymbol{\sigma}_n^z\boldsymbol{\sigma}_{n+1}^z\right), \tag{1}$$

where $\boldsymbol{\sigma}_n^\alpha$ $(\alpha = x, y, z)$ are the Pauli spin operators on site $n = 1, 2, \ldots, N$ (see Appendix A for definitions), satisfying periodic boundary conditions $\boldsymbol{\sigma}_{N+1}^\alpha = \boldsymbol{\sigma}_1^\alpha$, and $\{J_x, J_y, J_z\}$ are real coupling constants. In the most general case $(J_x \neq J_y) \neq J_z$, this model is known as the XYZ model. The special cases $(J_x = J_y) \neq J_z$ and $J_x = J_y = J_z = J$ correspond to the XXZ and XXX models, respectively. The eigenvectors and the eigenvalues of the completely isotropic XXX Hamiltonian were found in the pioneering work [2] by Bethe in 1931. In the ferromagnetic case $J > 0$, the Bethe ansatz equations provide an exact answer for the (trivial) ground-state properties and low-lying string-type excitations (an n-string is a group of n roots in the complex momentum plane distributed symmetrically and equidistantly around the real axis). In the antiferromagnetic case $J < 0$, the non-trivial ground state was constructed by Hultén [3]. He derived from the asymptotic limit $N \to \infty$ of Bethe's equations a linear integral equation for a particle distribution function in momentum space, the solution of which provides an explicit expression for the ground-state energy per site. More than 20 years later des Cloizeaux and Pearson [4] constructed excitations upon the antiferromagnetic ground state and found the asymptotic expression for their energy. The generalization of Bethe's method to the XXZ model, made by Yang and Yang [5, 6], was straightforward and brought the topic to a higher mathematical level. The exact solution of the XYZ model by Baxter in 1971 [7, 8, 9, 10] was a breakthrough. Baxter discovered a link between the quantum 1D XXZ and XYZ models and the equilibrium statistical mechanics of classical 2D six-vertex and eight-vertex models, respectively. He observed that the eigenstates of the transfer matrix of the six-vertex model are independent of one of the model parameters. Consequently, there exists an infinite family of commuting transfer matrices which originates from the so-called "Yang–Baxter equation" (or "star–triangle relation") fulfilled by the Boltzmann weights of the six-vertex model. The same observation holds also in the case of the eight-vertex model, for which Baxter obtained a system of Bethe-like transcendental equations. With the aid of these equations he was able to calculate the ground-state energy of the XYZ model and its critical properties which are non-universal in a weak sense: although the critical indices depend on the model's parameters, their ratios do not. The asymptotic energy of low-lying excitations of the XYZ model was obtained by Johnson, Krinsky and McCoy [11].

The fundamental property of integrable particle systems, possessing an infinite number of conservation laws, is the factorization property of multiparticle scattering into a sequence of two-particle scatterings. Two-particle scattering is elastic, i.e. not only the total momentum but also both individual particle momenta are conserved. In this context, the Yang–Baxter equation is the consistency condition for elements of the two-particle scattering matrix which ensures the invariance of three-particle (and, consequently, multiparticle) scattering with respect to the order in which two-particle scatterings are accomplished. The concept of the transfer matrix and the Yang–Baxter equation as the consistency condition played a central role in a program called the "Quantum Inverse-Scattering Method", established in late 1970s by Faddeev, Sklyanin, Takhtajan and their coworkers [12, 13]. The method is based on a relationship between integrable many-body models and integrable evolution equations [14, 15]. An important feature of the method, the algebraic construction of eigenstates of the transfer matrix [16, 17], gave an alternative name for it: the "algebraic Bethe ansatz". The systematic search for the solutions of the Yang–Baxter equation [18] resulted in the appearance of "Quantum Groups" [19, 20].

Another important group of integrable 1D models are non-relativistic continuum Fermi and Bose (the relationship between the spin and statistics is usually ignored) quantum gases with specific types of pairwise interactions. The crucial model was that of spinless (identical) bosons with attractive or repulsive δ-function interactions, initiated in 1963 by Lieb and Liniger [21, 22]. While the attractive bosons exhibit a collapse in the thermodynamic limit, the thermodynamic limit of the repulsive boson system is well behaved and the Bethe ansatz equations provide the ground-state (zero-temperature) properties as well as the energy of low-lying excitations. In 1969 Yang and Yang [23] derived from the Bethe equations the thermodynamic properties of repulsive δ-bosons at finite temperatures; this was the first exact treatment of thermodynamics for an interacting many-body system. The crucial observation was that also the holes, i.e. the unoccupied energy levels, contribute to the entropy of the system. Since the spectrum of excitations energies is relatively simple (the momenta are real, so there are only strings of length $n = 1$), the thermodynamics is determined by a coupled pair of integral equations for the distribution functions of the excitation energy and of the equilibrium particle (hole) densities in momentum space. The other spinless particle systems with integrable interactions, like the hard-core and inverse-square interactions, were treated analogously [24, 25].

The generalization of the Bethe ansatz method to systems of particles with internal degrees of freedom turned out to be complicated because in the scattering the internal states of the particles can be changed. The problem of spin-$\frac{1}{2}$ fermions with δ-interactions was solved in 1967 by Yang [26] and Gaudin [27] by using the "nested Bethe ansatz" and the Yang–Baxter equation as the consistency condition. The excited states of this model form strings of various lengths $n = 1, 2, \ldots$. The final result for the thermodynamics [28, 29, 30] is thus expressible in terms of the solution of an infinite set of coupled nonlinear integral equations, one for each string length n, known as the "thermodynamic Bethe ansatz" (TBA). These equations can be analyzed analytically only in special limits, e.g. in the zero and infinite limits of temperatures or interaction strengths. The same structure of the

TBA was observed in the case of the Heisenberg model [31, 32]. The strings can be avoided in a method developed by Destri and de Vega [33, 34] which leads to a single nonlinear integral equation.

The technique of the nested Bethe ansatz was applied to models of strongly correlated electrons in condensed-matter physics. The lattice version of the spin-$\frac{1}{2}$ fermion system with δ-interactions, the Hubbard model, was solved by Lieb and Wu in 1968 [35]. The exact solution showed the absence of a conducting–insulating Mott transition in one dimension. Anomalous scattering of a localized spin impurity with the conduction band at low temperatures leads to interesting phenomena known as the Kondo effects. The corresponding s-d exchange and Anderson models were solved by Andrei [36] and Wiegmann [37]. The 1D Luttinger model of interacting fermions [38] gave rise to bosonization techniques of Fermi operators [39]. Integrable models of BCS superconductors have been developing since the 1970s [40].

The Bethe ansatz technology was successfully applied also to integrable models of QFT in a (1+1)-dimensional spacetime, like the boson sine–Gordon model and its fermionic equivalent, the Thirring model [41], to obtain their exact scattering matrices and the mass spectrum [42, 43], the vacuum energy as a function of renormalized parameters of the theory [44], the relation between the coupling constant and the physical mass-scale [45], etc. Alyosha Zamolodchikov made a dominant contribution to this field.

As concerns the equilibrium statistical mechanics of classical systems, the first milestone occurred in 1944 when Onsager solved the 2D Ising model [46]. His exact solution showed the universality of critical phenomena and the fact that the critical indices in two dimensions are not mean-field like. Further lattice models of special interest were the vertex models, in which the state variables are localized on the edges connecting nearest-neighbor sites. Three cases of the six-vertex model – antiferroelectric F [47], ferroelectric KDP [48] and ice [49] – were solved by Lieb. The general case of the six-vertex model was solved by Sutherland [50]. The exact solution of the eight-vertex model by Baxter [7, 9] has already been mentioned in the context of the XYZ Heisenberg chain.

The statistical models presented above are defined on a regular discrete lattice structure. There exists another family of classical statistical models, the so-called fluids, formulated in continuum space. Concepts and methods used in the two fields are usually very different and the overlap between the physical communities is relatively small. While there exist many exactly solvable 2D lattice models, non-trivial fluid systems were solvable only in one dimension. A contribution of L.Š. and his coworkers consists of solving exactly the thermodynamics of the first continuum classical fluid in dimension higher than one: the 2D Coulomb gas of ± 1 point-like charges interacting via the logarithmic potential [51, 52]. The exact solution of a 2D classical Coulomb gas with charge asymmetry $+1, -\frac{1}{2}$ is also available [53].

There exist few monographs about the present subject. Those which we consider as the most relevant and therefore belong to our libraries are presented in chronological order in this paragraph. The famous book by Baxter [54] mainly concerns classical integrable models of equilibrium statistical mechanics. Gaudin summarizes his experience with the Bethe ansatz and the ground-state analysis in the technically rather difficult book [55]. Mattis's encyclopedia of exactly solved models in one dimension [56] contains over 80 reprinted papers with a short summary of each topic. The book by Korepin and Essler [57] contains reprinted articles in the field of condensed-matter physics. The Yang–Baxter equation, the general structure of its solutions and quantum groups are at the center of interest of the book [58]. Takahashi's book [59] is an encyclopedia of results about the thermodynamics of integrable many-body systems. Sutherland [60] and Kuramoto and Kato [61] concentrate on 1D models with interactions of inverse-square type. The 1D Hubbard model is reviewed in detail in the recent book [62].

A natural question arises: Why did we write another book about integrable systems? In our opinion, narrow specialization and the separation of communities is a feature of contemporary physics. Since we remember how many articles we had to find and to read in order to understand the subject in its many relevant aspects, we decided to write an extensive and at the same time self-contained course. We hope that, perhaps, this might help somebody to save time and to find new results in their own field. The main motivations for our (text)book are the following:

- The existing published books are usually oriented towards a restricted area of models and to specific methods. The present course encompasses all the important kinds of integrable models, including (1 + 1)-dimensional QFT and the classical 2D Coulomb gas, which, to our knowledge, have not been summarized in a book. Relatively complicated models, like the XYZ Heisenberg and general integrable spin-s quantum chains, are treated in detail as well.
- The mathematical level of some of the books is very high and requires a preliminary study of specific topics from the literature. The present course is self-contained, made mathematically as simple as possible. Derivations are complete, without any need to turn to original works. Only an elementary knowledge of quantum mechanics and equilibrium statistical physics is required. This makes the text accessible to graduate students in theoretical and mathematical physics.
- The methods and techniques presented in published books are usually traditional. We intend to include also modern trends in the TBA which are not included in standard textbooks. For example, the method of Destri and de Vega,

which avoids the usage of string roots of the Bethe equations in the derivation of thermodynamics at non-zero temperatures, is explained in detail. Another example is the TBA in QFT, as formulated by Alyosha Zamolodchikov in his derivation of the explicit relation between the Lagrangian parameter and the soliton mass for the sine–Gordon model.

- The course is not intended as an encyclopedia of the results obtained in the field of integrable systems. For each particular model, we give a detailed derivation of the Bethe ansatz equations, the specification of the ground state, the construction of the TBA and a discussion of the physical consequences which follow from the exact results.

The book is intended as a specialized textbook. Although the theory of integrable models is not a standard topic of basic undergraduate university courses, it is of importance for theoretically oriented graduate students. After a complete reading of the book, students should be able to understand original works in leading journals. Besides graduate students, the book is intended for specialists in integrable systems who would like to understand the application of the general quantum inverse-scattering method to other branches of physics, especially to QFT and the statistical mechanics of fluids, and potentially use special techniques in their own field. The textbook is also suitable for non-specialists, mathematical or theoretical physicists in many branches of physics, who would like to learn how to generate and solve an integrable many-body system.

The character and the aims of the book reflect our own experience in theoretical physics.

L.Š. is a leading researcher at the Institute of Physics of the Slovak Academy of Sciences in Bratislava, Slovakia. He is a specialist in the equilibrium statistical mechanics of lattice models and continuum fluids. Starting in 1991, he has occasionally taught graduate students at the Institute of Physics and at Comenius University in Bratislava in the field of statistical mechanics of integrable many-body systems. During his long-term stay (1993–1998) at the Courant Institute of Mathematical Sciences, New York University, he collaborated with Jerome K. Percus in the construction of exact density functionals for lattice models [63]. One of the topics of his special interest became classical and quantum, two-dimensional and higher-dimensional Coulomb fluids. This was just at the time of great discoveries in QFT in a (1+1)-dimensional spacetime. Being able to adopt the TBA techniques from the integrable sine–Gordon model, he contributed to equilibrium statistical mechanics by solving exactly the 2D Coulomb gas that was charge symmetric [51, 52] and with a charge asymmetry [53]. This was the first continuous fluid in dimension higher than one with exactly solvable thermodynamics. In 2001–2002, he was awarded a NATO fellowship in Laboratoire de Physique

Théorique, Université de Paris Sud in Orsay, to collaborate with Bernard Jancovici on Coulomb systems, mainly universal finite-size corrections [64] and the exact sum rules for the charge and density correlation functions [65]. This collaboration lasted up to 2010 and involved, e.g. the high-temperature aspects of the Casimir effect [66] and the fluctuations of the electromagnetic field at the interface between different electric media [67]. At present, he is collaborating with Emmanuel Trizac from Laboratoire de Physique Théorique et Modèles Statistique, Orsay, in the strong-coupling (low-temperature) description of classical Coulomb fluids based on Wigner crystallization [68].

Z.B. is a research professor in the Theoretical Physics Research Group of the Hungarian Academy of Sciences. Since his graduation he has been working on integrable models. He started his career by solving 2D conformal field theories with extended symmetries. Then his interest turned to the analysis of their integrable perturbations. He acquired knowledge of the bootstrap method designed to solve 2D integrable quantum field theories exactly. Using these techniques, with his collaborators they determined the exact spectrum of the boundary sine–Gordon theory [69]. In collaboration with Alyosha Zamolodchikov, they used the boundary TBA to investigate the sinh–Gordon theory on a finite interval and relate it to boundary Liouville theory [70]. He also developed methods to determine the form factors of operators localized both on integrable boundaries and defects [71, 72]. Recently, he has analyzed the finite-size spectrum of integrable quantum field theories. He developed a systematic expansion for the finite-size correction of the energy levels in various circumstances [73]. Exploiting the anti-de Sitter/conformal field theory correspondence he successfully applied the developed 2D integrable techniques to calculate the scaling dimensions of gauge-invariant operators in four-dimensional quantum field theories [74].

The material of this book is divided into five parts, with a short summary at the beginning of each part.

- In the first part, we deal with non-relativistic 1D continuum Fermi and Bose quantum gases of identical spinless particles.
- The second part is devoted to the description of the quantum inverse-scattering method and to the analysis of the related Yang–Baxter equation. We present the complete solution of spin-$\frac{1}{2}$ fermions with δ-interactions.
- The third part concerns integrable XXX, XXZ and XYZ Heisenberg models, with spin-$\frac{1}{2}$, and also isotropic models with general spin s. The thermodynamics is derived by using traditional methods based on the string hypothesis as well as by a simpler method of Destri and Vega which leads to a single nonlinear integral equation.

- The fourth part is devoted to systems of condensed-matter physics. We review the exact solutions of the Hubbard model. The exact solutions of the non-degenerate s-d exchange (Kondo) and Anderson models, describing the interaction of a single impurity with a conduction band, are worked out as well. The method of fermion bosonization is documented on the Luttinger many-fermion model. The integrable models of superconductors are presented.
- The fifth part concerns the complete solution of a relativistic (1+1)-dimensional integrable QFT, namely the sine–Gordon model. This model is first treated semi-classically, then its full quantum description is given. The relationship between the (1+1)-dimensional sine–Gordon model and the 2D classical Coulomb gas is explained and the exact thermodynamics of the latter model is derived.
- Appendix A describes an explicit construction of spin operators on a chain. The subject of Appendix B is the description of doubly periodic elliptic functions which are generalizations of the trigonometric functions in the complex plane.

Each part is divided into several chapters. Some exercises are presented at the end of each chapter. These exercises are intended either to avoid relatively simple algebraic calculations or to complement basic ideas in the main text. More complicated exercises are solved in detail; the solutions of simple exercises are only indicated.

L.Š. wrote the first four parts which concern integrable models of condensed-matter physics and equilibrium statistical mechanics. His writing is based on a series of lectures about integrable systems for graduate students given at the Institute of Physics and Comenius University in Bratislava and on a series of lectures at the Institute of Physics of the Czech Academy of Sciences in Prague. The fifth part, written by Z.B., is based on his lecture course delivered at Eötvös University in the fall semester of 2010. His aim was to present techniques and methods used to solve integrable QFT. The sine–Gordon model is a good pedagogical example in this respect as it is relatively simple, but contains all the essential ingredients one has to learn in order to solve more complicated integrable models.

This book is devoted to the memory of Alyosha Zamolodchikov. Z.B. had the honor of collaborating with this great magician in integrable QFT [70].

L.Š. is grateful to his teachers: Jerome K. Percus from the Courant Institute of Mathematical Sciences, New York University, and Bernard Jancovici from LPT, Université de Paris Sud, Orsay.

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