

Contents

<i>Preface</i>	<i>page xi</i>
1 Introduction	1
1.1 Historical considerations	1
1.1.1 Early results	1
1.1.2 Biography of Aleksandr Lyapunov	3
1.1.3 Lyapunov's contribution	4
1.1.4 The recent past	5
1.2 Outline of the book	6
1.3 Notations	8
2 The basics	10
2.1 The mathematical setup	10
2.2 One-dimensional maps	11
2.3 Oseledets theorem	12
2.3.1 Remarks	13
2.3.2 Oseledets splitting	15
2.3.3 "Typical perturbations" and time inversion	16
2.4 Simple examples	17
2.4.1 Stability of fixed points and periodic orbits	17
2.4.2 Stability of independent and driven systems	18
2.5 General properties	18
2.5.1 Deterministic vs. stochastic systems	18
2.5.2 Relationship with instabilities and chaos	19
2.5.3 Invariance	20
2.5.4 Volume contraction	21
2.5.5 Time parametrisation	22
2.5.6 Symmetries and zero Lyapunov exponents	24
2.5.7 Symplectic systems	26
3 Numerical methods	28
3.1 The largest Lyapunov exponent	28
3.2 Full spectrum: QR decomposition	29
3.2.1 Gram-Schmidt orthogonalisation	31
3.2.2 Householder reflections	31

Cambridge University Press

978-1-107-03042-8 - Lyapunov Exponents: A Tool to Explore Complex Dynamics

Arkady Pikovsky and Antonio Politi

Table of Contents

[More information](#)

vi

Contents

3.3	Continuous methods	33
3.4	Ensemble averages	35
3.5	Numerical errors	36
3.5.1	Orthogonalisation	37
3.5.2	Statistical error	38
3.5.3	Near degeneracies	40
3.6	Systems with discontinuities	43
3.6.1	Pulse-coupled oscillators	48
3.6.2	Colliding pendula	49
3.7	Lyapunov exponents from time series	50
4	Lyapunov vectors	54
4.1	Forward and backward Oseledets vectors	55
4.2	Covariant Lyapunov vectors and the dynamical algorithm	57
4.3	Dynamical algorithm: numerical implementation	59
4.4	Static algorithms	61
4.4.1	Wolfe-Samelson algorithm	62
4.4.2	Kuptsov-Parlitz algorithm	62
4.5	Vector orientation	63
4.6	Numerical examples	64
4.7	Further vectors	65
4.7.1	Bred vectors	66
4.7.2	Dual Lyapunov vectors	67
5	Fluctuations, finite-time and generalised exponents	70
5.1	Finite-time analysis	70
5.2	Generalised exponents	73
5.3	Gaussian approximation	77
5.4	Numerical methods	78
5.4.1	Quick tools	78
5.4.2	Weighted dynamics	79
5.5	Eigenvalues of evolution operators	80
5.6	Lyapunov exponents in terms of periodic orbits	84
5.7	Examples	89
5.7.1	Deviation from hyperbolicity	89
5.7.2	Weak chaos	90
5.7.3	Hénon map	94
5.7.4	Mixed dynamics	96
6	Dimensions and dynamical entropies	100
6.1	Lyapunov exponents and fractal dimensions	100
6.2	Lyapunov exponents and escape rate	103
6.3	Dynamical entropies	105

6.4	Generalised dimensions and entropies	107
6.4.1	Generalised Kaplan-Yorke formula	107
6.4.2	Generalised Pesin formula	109
7	Finite-amplitude exponents	110
7.1	Finite vs. infinitesimal perturbations	110
7.2	Computational issues	112
7.2.1	One-dimensional maps	114
7.3	Applications	115
8	Random systems	118
8.1	Products of random matrices	119
8.1.1	Weak disorder	119
8.1.2	Highly symmetric matrices	125
8.1.3	Sparse matrices	128
8.1.4	Polytomic noise	131
8.2	Linear stochastic systems and stochastic stability	136
8.2.1	First-order stochastic model	136
8.2.2	Noise-driven oscillator	137
8.2.3	Khasminskii theory	141
8.2.4	High-dimensional systems	142
8.3	Noisy nonlinear systems	146
8.3.1	LEs as eigenvalues and supersymmetry	146
8.3.2	Weak-noise limit	149
8.3.3	Synchronisation by common noise and random attractors	150
9	Coupled systems	152
9.1	Coupling sensitivity	152
9.1.1	Statistical theory and qualitative arguments	153
9.1.2	Avoided crossing of LEs and spacing statistics	157
9.1.3	A statistical-mechanics example	159
9.1.4	The zero exponent	160
9.2	Synchronisation	162
9.2.1	Complete synchronisation and transverse Lyapunov exponents	162
9.2.2	Clusters, the evaporation and the conditional Lyapunov exponent	163
9.2.3	Synchronisation on networks and master stability function	164
10	High-dimensional systems: general	168
10.1	Lyapunov density spectrum	168
10.1.1	Infinite systems	171
10.2	Chronotopic approach and entropy potential	173
10.3	Convective exponents and propagation phenomena	178
10.3.1	Mean-field approach	181

10.3.2	Relationship between convective exponents and chronotopic analysis	183
10.3.3	Damage spreading	185
10.4	Examples of high-dimensional systems	187
10.4.1	Hamiltonian systems	187
10.4.2	Differential-delay models	191
10.4.3	Long-range coupling	193
11	High-dimensional systems: Lyapunov vectors and finite-size effects	200
11.1	Lyapunov dynamics as a roughening process	200
11.1.1	Relationship with the KPZ equation	202
11.1.2	The bulk of the spectrum	207
11.2	Localisation of the Lyapunov vectors and coupling sensitivity	209
11.3	Macroscopic dynamics	213
11.3.1	From micro to macro	216
11.3.2	Hydrodynamic Lyapunov modes	218
11.4	Fluctuations of the Lyapunov exponents in space-time chaos	219
11.5	Open system approach	223
11.5.1	Lyapunov spectra of open systems	226
11.5.2	Scaling behaviour of the invariant measure	226
12	Applications	229
12.1	Anderson localisation	229
12.2	Billiards	231
12.3	Lyapunov exponents and transport coefficients	235
12.3.1	Escape rate	235
12.3.2	Molecular dynamics	236
12.4	Lagrangian coherent structures	237
12.5	Celestial mechanics	239
12.6	Quantum chaos	242
Appendix A	Reference models	245
A.1	Lumped systems: discrete time	245
A.2	Lumped systems: continuous time	246
A.3	Lattice systems: discrete time	247
A.4	Lattice systems: continuous time	248
A.5	Spatially continuous systems	249
A.6	Differential-delay systems	250
A.7	Global coupling: discrete time	250
A.8	Global coupling: continuous time	250
Appendix B	Pseudocodes	252

Appendix C Random matrices: some general formulas	256
C.1 Gaussian matrices: discrete time	256
C.2 Gaussian matrices: continuous time	257
Appendix D Symbolic encoding	258
<i>Bibliography</i>	259
<i>Index</i>	277