1 Introduction

1.1 Historical considerations

1.1.1 Early results

The problem of determining the stability of a given regime (e.g. the motion of the solar system) is as old as the concept of the dynamical system itself. As soon as scientists realised that physical processes could be described in terms of mathematical equations, they also understood the importance of assessing the stability of various dynamical regimes. It is thus no surprise that many eminent scientists, such as Euler, Lagrange, Poincaré and Lyapunov (to name a few), engaged themselves in properly defining the concept of stability. Lyapunov exponents are one of the major tools used to assess the (in)stability of a given regime. Within hard sciences, where there is a long-standing tradition of quantitative studies, Lyapunov exponents are naturally used in a large number of fields, such as astronomy, fluid dynamics, control theory, laser physics and chemical reactions. More recently, they started to be used also in disciplines, such as biology and sociology, where nowadays processes can be accurately monitored (e.g. the propagation of electric signals in neural cells and population dynamics).

The reader interested in a fairly accurate historical account of how stability has been progressively defined and quantified can refer to Leine (2010). Here, we limit ourselves to the recapitulation of a few basic facts, starting from the Galilean times, when E. Torricelli (1644) investigated the stability of a mechanical system and conjectured (in the modern language) that a point of minimal potential energy is a point of equilibrium.

Besides mechanical systems, floating bodies provide another environment where stability is naturally important, especially to avoid roll instability of vessels. Unsurprisingly, the first results came from a Flemish (S. Stevin) and a Dutch (Ch. Huygens) scientist: at that time, the cutting-edge technology of ship-building had been developed in the Dutch Republic. In particular, Huygens' approach was quite modern in that he addressed the problem by explicitly comparing two different states. D. Bernoulli too dealt with the problem of roll-stability, emphasising the importance of the restoring forces, which make the body return towards the equilibrium state. L. Euler was the first to distinguish between stable, unstable, and indifferent equilibria and suggested also the possibility of considering infinitely small perturbations.

The concept of stability was further developed by J.-L. Lagrange, who formalised the ideas expressed by Torricelli (for conservative dynamical systems), clarifying that, in the presence of a vanishing kinetic energy, the minimum of the potential energy corresponds to a stable equilibrium. The corresponding theorem is nowadays referred to as "Lagrange-Dirichlet" because of further improvements introduced by J. P. G. L. Dirichlet.

In the nineteenth century, fluid dynamics provided many examples where the stability assessment was far from trivial. Some scientists (notably Lord Kelvin) were striving to unify physics under the paradigm of the motion of perfect liquids, and such an approach required the stability of various forms of motion. At a macroscopic level, in the attempt of predicting the Earth's shape, the problem of determining the stable shape of a rotating fluid, under the influence of the sole action of centrifugal and (internal) gravitational forces, was posed. The studies led to the conclusion that, in some conditions, ellipsoidal shapes are to be expected, but the problem was not fully solved (see Section 1.1.2 on Lyapunov's biography).

On a more microscopic level, hydrodynamics proved to be an extremely fertile field for the appearance of instabilities: concepts such as sensibility to infinitesimal and finite perturbations were present in the minds of esteemed scientists. G. Stokes was one of the pioneers: he stipulated that instabilities naturally occur in the presence of rapidly diverging flow lines, such as past a solid obstacle. Slightly later, H. Helmholtz and W. Thomson discovered that the surface separating two adjacent flows may lose its flatness. Contrary to the instability foreseen by Stokes, which was based only on conjectures, the latter one, nowadays referred to as the Kelvin-Helmholtz instability, was also derived directly from the hydrodynamics equations. Last but not least, Lord Kelvin strived to develop a vortex theory of matter, which, however, required the stability of the underlying dynamical regimes. Only at the end of his career did he convince himself that his ideas were severely undermined by the unavoidable presence of instabilities. The interested reader can look at the exhaustive review by Darrigol (2002).

Celestial mechanics proved to be another fruitful environment for the development of new ideas. In order to appreciate how relevant the subject was in those times, it is sufficient to mention that when P. S. Laplace studied perturbatively the behaviour of three gravitationally interacting particles (the so-called 3-body problem), he referred to it as to the "world system". Heavily relying on recent results by Lagrange, Laplace concluded that the semi-major axis of the orbits is characterised by periodic oscillations. Thus, he concluded in favour of stability, meaning that the fluctuations are bounded. A bit later, S. D. Poisson discovered that second- and third-order terms generate a secular contribution of the type $At \sin \alpha t$; however, as remarked by C. G. J. Jacobi, it was not clear whether such a contribution would survive a higher-order analysis. All in all, no clear answer had yet been given by the end of the nineteenth century. This is the reason why King Oscar II of Sweden decided to offer a prize for those who could find an explicit solution. H. Poincaré won the prize even though he did not actually solve the problem. On the contrary, his work established the existence of unavoidable high sensitivity to initial conditions: what was later called the 'butterfly effect' by the metereologist E. N. Lorenz.[1] Poincaré received the

[1] The expression 'butterfly effect' was arguably introduced by Lorenz in 1972, when he gave a talk at the American Association for the Advancement of Science entitled "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

prize for the revolutionary methods that he developed to gain insight about the behaviour of generic dynamical systems.

A last environment where stability turned out to be of primary importance is related to engineering applications. In the nineteenth century, with the advent of steam engines, it became necessary to regulate the internal pressure inside the boiler. This problem represented the starting point for the birth of a new discipline: automatic control theory. J. C. Maxwell analysed the stability of Watt's flyball regulator by linearising the equations of motion. Independently, I. A. Vyshnegradtsky used a similar approach to study the same problem in greater detail.

1.1.2 Biography of Aleksandr Lyapunov

Here, we briefly summarise some basic facts of the biography of Aleksandr Mikhailovich Lyapunov, mostly relying on Smirnov (1992) and Shcherbakov (1992).

Aleksandr Lyapunov was born in 1857 in Yaroslavl. After completing his gymnasium studies in Nizhny Novgorod, Lyapunov moved to the University of St. Petersburg, where the Mathematical Department was blooming under the direction of Pafnuty Chebyshev, who soon became the supervisor of his graduate studies. Chebyshev used to say that "every young scholar . . . should test his strength on some serious theoretical questions presenting known difficulties". As a matter of fact, Lyapunov got involved in a problem that had been earlier proposed to other students (he discovered this later in his career), namely that of determining the shape of a rotating fluid. As his efforts proved unsuccessful, Lyapunov

Fig. 1.1 A. M. Lyapunov in 1902, in Kharkov. Photo courtesy of Elena Alexeevna Lyapunova.

decided to refocus his work, preparing a dissertation entitled *On the stability of elliptic forms of equilibrium of rotating fluids*, which nevertheless allowed him to be awarded a Master's degree in applied mathematics (1884) and made him known in Europe. In 1885, Lyapunov was appointed Privatdozent in Kharkov, where he worked on the stability of mechanical systems. His main results were summarised in a remarkable thesis entitled *The general problem of the stability of motion*, which granted him a PhD at Moscow University (1892). The dissertation contains an extraordinarily deep and general analysis of systems with a finite number of degrees of freedom. Interestingly, Lyapunov mentioned H. Poincaré as one of his principal sources of inspiration.

In 1893, Lyapunov was promoted to ordinary professor in Kharkov. In the following years, he kept studying stability properties of dynamical systems, investigated the Dirchlet problem, and engaged himself in problems of probability theory, contributing to the central limit theorem and paving the way to the rigorous results obtained by his friend Andrei Markov. In 1901 he became head of the department of Applied Mathematics at the Russian Academy of Sciences in St. Petersburg (the position, without teaching duties, had been vacant since 1894, when Chebyshev died).

After having completed a cycle of papers on the stability of motion, Lyapunov came back to the question posed to him by Chebyshev about 20 years before and much related to the problem of determining the form of celestial bodies, earlier formulated by Laplace. While he was still struggling to find a solution, Lyapunov became aware of a book published by Poincaré in 1902 on the same problem and managed to acquire a copy. From a letter sent by Lyapunov to his disciple and close friend Steklov: "To my greatest surprise, I did not find anything significant in this book . . . Thus my work has not suffered and I apply myself to it afresh". The book by Poincaré essentially contained previous (known) concepts with little advancements.

Shortly after, the astronomer George Darwin (son of Charles Darwin) published some papers on the same subject, concluding that pear-shaped forms are to be expected. Lyapunov completed his studies in 1905: a treatise of about 1000 pages, with some mathematical calculations made up to 14 digits when necessary. He indeed discovered deviations from ellipsoids, but he also showed that pear-shaped forms are unstable. The controversy with Darwin went on for some years, until it was eventually settled in 1917, when another British astronomer, J. H. Jeans, confirmed that Lyapunov was right.

In 1917 Lyapunov left St. Petersburg for Odessa, so that his wife could receive treatment for tuberculosis. On the day of his wife's death, Aleksandr Lyapunov committed suicide.

1.1.3 Lyapunov's contribution

The first formal definition of stability was given by Lyapunov in his PhD thesis: a given trajectory is stable if, for an arbitrary ε, there always exists a δ such that all other trajectories starting in a δ-neighbourhood of the given one remain at most at a distance ε to it. He introduced also what was later called asymptotic stability, to refer to cases where sufficiently small perturbations eventually die.

Lyapunov introduced also two methods to assess the stability of a given solution. The first method was based on a "standard" perturbative analysis; he was very interested in identifying those cases where it is necessary to go beyond the first order to characterise correctly the perturbation dynamics. As a result, he introduced the "characteristic number" λ_L. It is basically the opposite of what is nowadays called the (characteristic) Lyapunov exponent. In fact, he defined λ_L as the exponential rate which has to be added to balance the growth rate of a given perturbation $\delta(t)$: in other words, assuming that $\delta(t) \approx e^{\lambda t}$, he would define the characteristic number λ_L as the value such that $\delta(t)e^{\lambda_L t}$ neither diverges nor converges exponentially.

The second, or direct method, deals with the introduction of a pseudo-energy function (nowadays called Lyapunov function) that vanishes in the equilibrium point and is otherwise positive, and decreases (or does not increase) along a generic trajectory. This is an extension of the ideas of Torricelli and Lagrange to a context where the potential energy is not defined a priori. Lyapunov's PhD thesis was translated into French in 1908, while one had to wait until 1992 to see the first English translation (Lyapunov, 1992).

Although Lyapunov himself attached more importance to the first method, he became famous for the second method. Nevertheless, even within Russia, the first practical applications of his stability theory were not made until the 1930s by N. G. Chetayev and I. G. Malkin at the Kazan Aviation Institute. The reader interested in the development of the second method is invited to consult Parks (1992).

1.1.4 The recent past

Although Lyapunov exponents (LEs) were formally introduced at the end of the nineteenth century, for a long time, they did not attract the attention of scientists. One reason is that most efforts were initially devoted to characterising the stability of either constant, or periodic dynamics, in which case the problem reduces to determining the eigenvalues of a suitable matrix (see Chapter 2).

A second reason why the application of the Lyapunov exponents was so much delayed with respect to the time of their definition was the difficulty of dealing with noncommuting entities. In fact, as shown in Chapter 2, the LEs are generated by multiplying (infinitely many) matrices. The first analytical result was obtained by Furstenberg and Kesten (1960), who basically proved the existence of the maximum and the minimum Lyapunov exponent. The full multiplicative ergodic theorem, ensuring the existence of as many Lyapunov exponents as the dimension of the space where the matrices operate, was proved later by Oseledets (1968) under fairly general conditions.

A further reason for the prolonged lack of specific studies of the Lyapunov exponents was the lack of workable instances of what was later defined as chaotic dynamics; in other words it was not clear which trajectories to consider for the underlying linearisation. It was only after the advent of the electronic computer that (approximate) trajectories could be generated and thereby characterised. The reception of the first physical model of a chaotic dynamical system, the Lorenz attractor (Lorenz, 1963), provides an enlightening

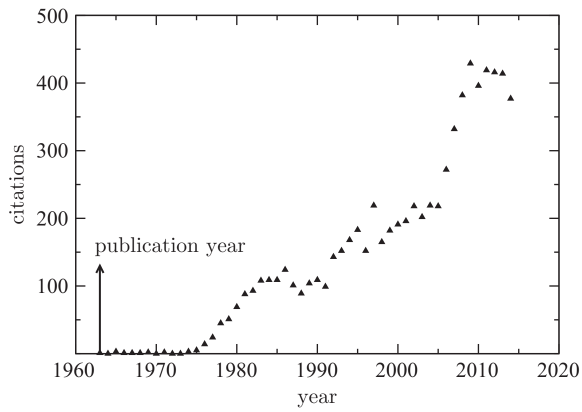

Fig. 1.2 Number of citations per year of the seminal paper (Lorenz, 1963), starting from the year of its publication. (Data from ISI Web of Science (www.wokinfo.com).)

view. In Fig. 1.2, one can see that the number of yearly citations of such a seminal paper remained pretty low until the late 1970s, when it started to explode. That was the time when computers became generally available to scientists, allowing them to work "experimentally" on chaotic dynamics. It is, in fact, in the same period that the relevant algorithms for the Lyapunov-exponent computation were developed (Shimada and Nagashima, 1979; Benettin et al., 1980a, b).

1.2 Outline of the book

Writing a book requires organizing a set of items in a sequential way, but in many cases, such as the present one, several interconnections are present, which make any ordering inconvenient. Therefore, we find it useful to summarise the book's content here, highlighting the connections among the various chapters.

Chapters 2 and 3 are devoted to the introduction of the main general properties of Lyapunov exponents and to the related numerical tools; the information contained therein will suffice for those interested in a basic Lyapunov analysis. The pseudocodes described in Appendix B provide further guidance for the implementation of the required algorithms. More precisely, the proper mathematical setups (continuous vs. discrete time) are introduced in Chapter 2, where they are followed by a discussion of the main properties, which include the effect of symmetries and the invariance of the LEs under changes of coordinates. In Chapter 3, various algorithms for the computation of LEs are introduced; they include the standard methods based on vector orthogonalisation at finite time intervals, as well as continuous methods. A specific section is devoted to models characterised by discontinuities in phase space where some care is required. The various sources of errors are also briefly discussed.

In nonlinear dynamical systems, there exists a relationship between LEs and various properties of their invariant measures, such as fractal dimensions and dynamical entropies. This connection is discussed in Chapter 6, where the Kaplan-Yorke and Pesin formulas are reviewed.

When computed over finite time, LEs naturally exhibit fluctuations that can be described via two different, but equivalent, methods. The first one, based on the computation of suitable moments, leads to the definition of the so-called generalised LEs. The second one is instead based on large-deviation theory. Both are described in Chapter 5, where we also illustrate a powerful numerical technique for the detection of rare fluctuations (see Section 5.4.2). Some generalised LEs are theoretically more manageable than the usual LEs; they can be used to derive approximate analytic expressions (some examples are presented in Chapter 8). In Chapter 6, we discuss the implication of LE fluctuations on the definition of the family of (generalised) fractal dimensions and entropies. The size of the fluctuations proves particularly useful to identify those borderline cases, where positive and negative finite-time LEs coexist. Finally, fluctuations arise also in spatially extended systems, where it is important to understand their scaling behaviour with the system size (this is discussed in Chapter 11).

If finite, instead of infinitesimal, perturbations are considered, the so-called finite-amplitude LEs can be defined. The usefulness and the limits of this generalisation are discussed in Chapter 7. Of particular relevance is the case of collective chaos, where such exponents allow identification of the presence of macroscopic instabilities.

LEs quantify the growth rates of generic (infinitesimal) perturbations. Further information is contained in the direction of the perturbations, i.e., in the orientation of the corresponding Lyapunov vectors. This issue is extensively discussed in Chapter 4, where different definitions and algorithms for their reconstruction are presented. The structure of Lyapunov vectors is discussed also in Chapter 11, where an analogy with rough interfaces (in spatially extended systems) allows for a fairly complete characterisation. An important property that is worth testing in high-dimensional contexts is the extended vs. localised nature of the Lyapunov vectors. The presence of localised vectors (especially the first one) helps to develop approximate expressions for the LE; some examples are discussed in Chapters 8 and 10. Furthermore, the presence of extended vectors is conjectured to be a signature of the presence of collective chaos (see Chapter 11).

Exact analytical expressions for the LEs are not typically available in generic nonlinear/disordered systems. Some results are obtained with the help of suitable perturbative techniques when the fluctuations are quite small. Other results can be derived under the assumption of relatively simple statistical properties (e.g. lack of temporal and/or spatial correlations). The corresponding theories are reviewed in Chapter 8. This chapter is particularly important for those interested in applications of LEs to intrinsically random situations, but it also provides a basis to obtain approximate results for chaotic dynamical systems.

Chapters 9, 10 and 11 deal with LEs in complex systems. In Chapter 9, we start from the relatively simple setup of two coupled systems, used as a testbed for understanding the effect of coupling on nearly identical units (e.g. coupling sensitivity and synchronisation). Chapter 10 is devoted to a general discussion of high-dimensional systems: various classes of setups are considered, but the emphasis is mostly put on systems characterised by short-range interactions in a one-dimensional physical space. One of the key properties is the extensivity of the chaotic dynamics (i.e. the proportionality between the number of active degrees of freedom and the dimension of the phase space), which manifests itself as a well-defined Lyapunov density spectrum (in the so-called thermodynamic limit). Finally, Lyapunov vectors in spatially extended systems are analysed in Chapter 11, mostly through an insightful analogy with rough interfaces. Furthermore, models characterised by a chaotic collective behaviour are used to elucidate differences (and possible connections) between the stability analysis at a microscopic and a macroscopic level.

In Chapter 12 we review several physical problems, where the application of Lyapunov exponents plays an essential role. The four appendices contain some technical details. In Appendix A, almost all models that are used as a reference across the book are properly defined. Appendix B contains the most basic pseudocodes to be used for the computation of Lyapunov exponents and vectors. In Appendix C, we derive some general formulas used in Chapter 8 for the computation of LEs in products of random matrices. Finally, rudiments of symbolic encoding are presented in Appendix D: they prove useful in implementing some techniques introduced in Chapter 5.

1.3 Notations

While writing a monograph, the use of homogeneous notations is a must to avoid unpleasant misunderstandings on the meaning of some symbols. We have made our best efforts to avoid overlaps, but, given the large number of quantities we had to introduce, this has not been always possible. However, we have tried to clearly identify and differentiate the general observables, such as time, space and phase-space variables. Particular care has been taken in differentiating the many different types of Lyapunov exponents that are discussed in the book. The main notations are summarised in the following tables.

Acronyms

Abbreviation	Description
BV	Bred vectors
CLV	Covariant Lyapunov vector
FAE	Finite amplitude exponent
FLI	Fast Lyapunov indicator
FPU	Fermi-Pasta-Ulam
GALI	Generalised alignment index
GS	Gram-Schmidt orthogonalisation
KPZ	Kardar-Parisi-Zhang
LE	Lyapunov exponent
SVD	Singular value decomposition

General notation

Symbols used	Description	Page
$U, V, \mathbf{U}, \mathbf{V}$	Capital letters are used to refer to variables in phase space; bold letters denote vectors of variables.	10, 11
$u, v, \mathbf{u}, \mathbf{v}$	Lower-case letters refer to variables in tangent space obeying linearised equations; bold letters denote vectors in tangent space.	10
$F, \mathbf{F}$	Capital F usually denotes the velocity field for a differential equation or the map function; bold denotes a system of equations or a high-dimensional map.	10, 11
t	Time (either discrete or continuous)	10, 11
$\mathsf{J}, \mathsf{H}, \mathsf{M}$	Sans serif letters denote matrices	
$\mathsf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$	Instantaneous Jacobian of a map	10
K	Instantaneous Jacobian for continuous time differential equations	11
H	The product of many Js for a sequence of Jacobian matrices or the integration of K over a finite time for continuous time case	10
x	Spatial variable (discrete or continuous) for spatio-temporal dynamics, written as an index, e.g., U_x, when discrete	166, 172, 173, 247
L	System size for spatially extended systems	168

Different types of Lyapunov exponents

λ_k	kth Lyapunov exponent (typically ordered from the largest to the smallest one)	12
$\mathcal{S}_n$	Sum of the first n Lyapunov exponents (or the sum of all Lyapunov exponents)	21, 22
$\Lambda(t_0, \tau)$	Finite time Lyapunov exponent determining the growth rate in the time interval $t_0 < t < t_0 + \tau$	39, 71
$\Gamma = \Lambda\tau$	Overall expansion factor over time interval τ	39, 71
$\mu = \exp[\lambda]$	Multipliers (i.e. their logarithms are the Lyapunov exponents)	12
$\mathcal{L}(q)$	Generalised Lyapunov exponents	74
$G(q) = \mathcal{L}(q)q$	Characteristic function of perturbation growth	74
ℓ	Finite amplitude exponent (averaged)	111
$\mathbb{L}$	Instantaneous finite amplitude exponent (related to ℓ, like Λ to λ)	111
$\mathscr{L}$	Convective (velocity-dependent) exponent	179

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Excerpt
More information

2 The basics

In this chapter we introduce the mathematical background that is necessary for defining the Lyapunov exponents (LEs). We then introduce the LEs by referring to the Oseledets multiplicative ergodic theorem. As everywhere else in the book, rather than presenting fully rigorous derivations, we prefer to propose physical and heuristic arguments with the help of examples of increasing complexity.

General properties of the LEs are also illustrated to help explain the role of LEs in general physical contexts.

2.1 The mathematical setup

The notion of Lyapunov exponents emerges while assessing the stability of generic trajectories of a dynamical system. Two classes are typically studied in the scientific literature, namely discrete- and continuous-time models.

A discrete-time dynamical system, usually referred to as a map, is represented by the recursive relation

$$\mathbf{U}(t+1) = \mathbf{F}(\mathbf{U}(t)). \tag{2.1}$$

Here, $\mathbf{U}$ is an N-dimensional state variable, t is an integer variable denoting time and $\mathbf{F}(\mathbf{U})$ is a (possibly non-invertible) function from $\mathbb{R}^N$ to $\mathbb{R}^N$. An initial condition $\mathbf{U}(0)$ uniquely defines the trajectory $\mathbf{U}(t)$, which may be generated by iterating the relation (2.1), and is assumed to be well defined for all $t > 0$. Its *stability* can be assessed by selecting a second close trajectory and thereby checking whether it remains close to the original one. It is customary to consider infinitesimal perturbations. This requires linearising the map (2.1) (this step assumes a sufficient smoothness of the map). The perturbation $\mathbf{u}(t)$ of the trajectory $\mathbf{U}(t)$ follows the linear transformation

$$\mathbf{u}(t+1) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t)\mathbf{u}(t) =: \mathsf{J}(t)\mathbf{u}(t), \tag{2.2}$$

where J is the so-called Jacobian matrix. The current value $\mathbf{u}(t)$ of the perturbation is generated by iterating Eq. (2.2),

$$\begin{aligned}\mathbf{u}(t) &= \mathsf{J}(t-1)\mathbf{u}(t-1) = \mathsf{J}(t-1)\mathsf{J}(t-2)\mathbf{u}(t-2) = \dots \\ &= \prod_{k=0}^{t-1} \mathsf{J}(k)\mathbf{u}(0) = \mathsf{H}(t)\mathbf{u}(0),\end{aligned} \tag{2.3}$$