Introduction

The idea that mathematics is reducible to logic has a long history. But it was Frege who gave logicism an articulation and defense that transformed it into a distinctive philosophical thesis of major consequence. With Frege's intervention, the doctrine came to have a profound influence on the development of philosophy in the twentieth century: it led directly to modern second-order logic – essentially the product of Frege's analysis of the concept of generality – and to notions of analysis that came to define philosophical thinking in Britain, on the continent, and in America.

The recent revival of interest in Frege's philosophy of arithmetic has shown it to be of much more than just historical interest. This revival was occasioned by Crispin Wright's important discovery that a great deal of Frege's theory of number is unaffected by the contradiction in his theory of classes. The reevaluation of Frege inspired by Wright's discovery has spawned a large philosophical and technical literature. Though informed by the technical results recorded in this literature, the goal of the essays collected here is general and philosophical. In addition to their contribution to the reevaluation of Frege's philosophy of mathematics, they argue that Frege's work contains insights that bear on fundamental problems of interpretation that arise in the context of empirical theories. The essays offer a series of critical reflections on a variety of applications of different conceptions of analysis – all of them having their source in the logicist tradition – to the philosophy of mathematics and the philosophy of science. An assumption they attempt to encourage is that there is a great deal still to be learned about the nature of analysis and how it was practiced by Frege and the principal historical figures in the logicist tradition who followed him – Russell, Ramsey, and Carnap.

While I hope that the essays in this volume will be of interest to historians, especially historians of the twentieth-century analytic tradition, they are not – nor do they pretend to be – scholarly historical studies. I have tried to exhibit the main features of the variety of alternative analytical approaches that different logicists have pursued in their separate attempts to define what is distinctive about arithmetical knowledge. And I have been especially concerned to isolate what was characteristic of Frege's contribution to the analysis of our knowledge of number, and to illuminate, by comparison with his contribution, the analyses of others. But I have not hesitated to emphasize a reconstruction of an author's view if I thought doing so was clarifying or suggestive of a more successful approach to the problem I was considering; I have proceeded in this way even when such a reconstruction represented an obvious departure from the view that inspired it. This is particularly true of my accounts of Frege's analysis of arithmetical knowledge in Chapter 1, Carnap's elaboration and defense of the thesis that mathematics is not factual in Chapter 2, and *Principia*'s theory of functions and classes in Chapter 10.

Frege's criterion of identity for number occupies a prominent position in these essays. The criterion was put forward in *Grundlagen* as an "informative" answer to the question, 'Under what conditions should we regard statements of the form, "The number of *F*s is the same as the number of *G*s," true?' According to Frege, such statements express "recognition judgments": they express our recognition that the same number has been presented in two different ways, as the number of one or another concept. The criterion of identity asserts that a recognition judgment is true if and only if its constituent concepts are in one–one correspondence.

In the secondary literature, Frege's criterion of identity is sometimes referred to as a partial contextual definition of the truth conditions for numerical statements of identity. Setting aside the notion of a contextual definition, the definition the criterion expresses is said to be *partial* because it covers only some identities involving numerical singular terms. It covers those cases where the singular terms are of the form 'the number of . . .,' or, as with numeral names, are definable in terms of expressions of this form, but it leaves out cases where at least one of the singular terms is not of this character. However, even if the criterion of identity is in this sense only a partial contextual definition, when taken as an axiom or principle, it suffices for a mathematically adequate theory of number – one that gives rise to an analysis of arithmetical knowledge of considerable philosophical interest.

In the essays that follow, Frege's criterion of identity is discussed in connection with two rather different applications of it. It is discussed first in Chapter 1 in connection with the theory of arithmetical knowledge, where I formulate a theory that is *close* to Frege's intended account and compare it with the theory of Whitehead and Russell. In Chapter 2 the criterion is discussed in connection with Carnap, where it is applied to the problem of distinguishing between formal and factual components of the language of science. The essays of both chapters are new to this volume, but since the second application of Frege's criterion of identity is likely to be a much less familiar kind of application of the criterion than the first, it might be worthwhile to comment further on it here.

Hume had the idea that the difference between the epistemic status of arithmetic and geometry can be traced to the different criteria of identity that govern their fundamental notions – the notions of number and length, respectively. Indeed, when in sect. 63 of *Grundlagen*, Frege introduces the criterion of identity, he quotes from Hume's *Treatise*:

> We are possest of a precise standard, by which we can judge of the equality and proportion of two numbers; and according as they correspond or not to that standard, we determine their relations without any possibility of error. When two numbers are so combin'd as that one has always an unite answering to every unite of the other, we pronounce them equal[.] (Book I, Part III, sect. I, para. 5)

That Frege quotes Hume in the course of introducing his criterion of identity for number led George Boolos to coin the name 'Hume's principle' for the criterion of identity. Motivated in no small measure by the attractiveness of Boolos's essays on the topic, the name has become established in the secondary literature devoted to the reevaluation and revival of Frege's contributions to the philosophy of mathematics. Whether or not it was intended, Boolos's terminology suggests an important change in perspective: it focuses attention on the role of the criterion of identity as an independently motivated axiom, rather than a definition that is incomplete or partial.

It is interesting to note that Frege does not quote Hume in full, but leaves out the remark with which the passage concludes:

> and 'tis for want of such a standard of equality in extension, that geometry can scarce be esteem'd a perfect and infallible science.

Frege had little interest in Hume's application of criteria of identity – "standards of equality" in Hume's phrase – to the problem of elucidating the epistemological differences between arithmetic and geometry. But he certainly hoped to gain acceptance for Hume's principle, given its pivotal position in the central mathematical argument of *Grundlagen*: the entire development of the theory of number is based on it, and Frege may well have supposed that by citing the precedent of Hume, he could enhance the plausibility of basing his theory on this criterion. Nevertheless, Frege's favorable citation of Hume obscures important differences in their understanding of the criterion of identity: it is clear from the passage Frege quotes that Hume conflates the notion of a class with that of a plurality, and that he treats a plurality of things as a "number," rather than treating numbers as separate from – and representative of – the concepts or classes to which they "belong." Each of these differences points to a characteristic that is a hallmark of Frege's philosophy of mathematics.

The context of Hume's introduction of the standard of equality for number is actually *geometric*, and arithmetic enters the discussion only coincidentally. Hume observed that the idea that space is infinitely divisible cannot be based on perception or imagination since reflection quickly reveals that any division we effect must terminate after some indefinite but finite number of steps. The terminus of any process of division is not susceptible to further division, and although the termini are in this sense unextended spatial minima, they are minima that are reached after finitely many divisions. Against the suggestion that two spatial intervals are equal in length if they are composed of the same number of spatial minima, Hume argues that our idea of an interval is always of one for which the minima are "confounded" with one another. This confounding has the consequence that it is impossible to base a notion of sameness of length on a one–one correspondence between the minima belonging to the two intervals. The situation is altogether different in the case of our judgments of sameness of number, for these concern pluralities of "unites" which we can clearly grasp as separate from one another. Hence the question whether the units of the pluralities to which arithmetic is applied are or are not in one–one correspondence has a clear sense and a determinate answer. But in the case of geometry, we must be content with the usual criterion for sameness of length, namely the coincidence of the end-points of the intervals being compared: in other words, the geometrical situation is just as it would be if we were dealing with a continuum, where comparisons are necessarily approximate. Geometry is therefore fallible and "imperfect" in a way that arithmetic is not.[1]

In Chapter 2 I argue that there is a sense in which the question, 'What distinguishes applied geometry from applied arithmetic?' is illuminated by an examination of the criteria of identity that are appropriate to arithmetical and geometric notions. Although Hume must be credited for first raising

1 Notice that the appropriateness of the arithmetical criterion in the context of a discrete space can be reasonably questioned even if one does not accept the relevance of Hume's contention that spatial minima are necessarily "confounded" by us. It would not be unreasonable to reject the arithmetical criterion for sameness of length if the judgments of congruence to which it led conflicted with those that are forced by our customary geometric procedures for establishing congruence.

the possibility that differences associated with their criteria of identity bear on the difference in epistemic status we attribute to applied geometry and applied arithmetic, his positive proposals fail to isolate the correct point. Contrary to Hume, the difference is not that one standard is precise in a way that the other can never be, or that one enjoys a kind of infallibility that the other can never attain. Rather the central difference is that the criteria of identity for geometrical notions like *length* are empirically constrained in a way that the criterion of identity for *number* is not. The manner in which geometrical notions are differently constrained depends on methodological considerations of some subtlety, requiring for their full appreciation the discussion of chronometry and the criterion of identity for *time of occurrence*. The chapter concludes by showing how these observations can be exploited to support Carnap's claim that applied geometry has a factual content that applied arithmetic lacks. This claim lies at the center of Carnap's insistence that an adequate reconstruction of physics must include a principled distinction between its factual and nonfactual components.

There is an interesting conceptual difference between Frege and Hume which would have been clear to Frege, but would have fallen outside Hume's conceptual horizon. This difference is attributable to their divergent views on the scope of logic. Although a small point, it leads naturally to an observation about Russell and a major preoccupation of several of the essays collected here. Hume's discussion of infinite divisibility raises two questions: (i) 'What is the basis for our belief that space is infinitely divisible?' and (ii) 'How do we come to the concept of infinite divisibility?' Hume treats the first question as one about space as we perceive or imagine it; he then proceeds to address the question on the basis of the phenomenological evidence that is provided by our perception and imagination. As for the second question, Frege had the conceptual resources from which he could argue that whether or not the basis for our *belief* in infinite divisibility is decided by what is allowed by our perceptual and imaginative capacities, our *concept* of infinite divisibility does not rest on perception or imagination *because infinite divisibility – i.e. denseness – is a purely logical notion*. Even if Hume were to agree with the conclusion that our concept of infinite divisibility does not rest on our perception or imagination, he could not have advanced this argument on its behalf.

Similarly, independently of whether the basic laws of arithmetic are reducible to the laws of logic, it was a discovery of some interest that the truth of a recognition judgment should depend on a condition that is characterizable in terms drawn exclusively from logic. That this notion of similarity of properties or Fregean concepts is a wholly logical one, and therefore a notion of similarity that does not depend on experience or "a priori intuition," was the insight that led Russell to the view that the notion of *structural similarity* – which is a simple extension of one–one correspondence – might be capable of resolving classical metaphysical and epistemological problems about the relation of appearance to reality. The origins of Russell's "structuralism" therefore trace back to ideas that were integral to the development of logicism. Chapter 4 contains an extended introduction to the essays that deal with Russell's structuralism and its elaboration in the work of Ramsey, Carnap, and others, and its initial sections should be consulted for an overview of the topics these essays address.

Although the order of the chapters is the order in which I believe the essays are best read, each essay stands by itself and can be read independently of the others. I should, however, note that the previously unpublished essays were written not only to advance new ideas, but to introduce and orient the reader to the older essays in this collection. Indeed, the first four essays have a particularly strong claim to being read before the others and in the order in which they are presented, and they come close to forming a monograph of their own. As I mentioned earlier, Chapter 1 refines my view of what is of contemporary interest in Frege's theory of number and presents the background to Chapter 2; Chapter 1 also serves as an introduction to all the essays that are specifically concerned with logicism and the philosophy of mathematics, namely Chapters 8–11. Chapter 2 is the chapter most explicitly concerned with applying the central ideas of Frege's logicism to issues in the philosophy of science, and it forms a natural transition from Chapter 1 to Chapters 3 and 4. Chapter 3 addresses whether someone sympathetic to Carnap's views on questions of realism in the philosophy of mathematics is committed to a similar view of such questions in the philosophy of physics. In particular, it considers whether his distinction between internal and external questions can be applied differentially to questions about the existence of abstract entities and questions about the existence of the theoretical entities of physics. Chapter 4 is not only an introduction to Chapters 5–7; it also connects the discussion of Carnap's distinction between internal and external questions with issues raised by "structuralist" theories of theoretical knowledge, and it complements and extends the earlier discussion in Chapter 3 of the reality of theoretical entities.

In general, I have made only stylistic changes or changes for the purpose of clarification to the previously published essays. The major exception is Chapter 10 which, in addition to many such changes, has been significantly expanded by the addition of an appendix on Russell's propositional paradox, its extension to Fregean thoughts, and its relevance to Dedekind's theory of our knowledge of number. Where I have found a formulation or point of view with which I no longer agree, I have, in cases where it seemed important to do so, indicated this by a new footnote; but I have refrained from rewriting passages to reflect my current view. All footnotes that record such disagreements or direct the reader to a subsequent discussion use a special mark and are annotated as having been added in 2012.

CHAPTER I

Frege's analysis of arithmetical knowledge

> Philosophy confines itself to universal concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents *a priori*, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object thus constructed.
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> (Kant 1787, A716/B744)

Frege most directly engages Kant when, in *Grundlagen*,[1] he presents his formulations of the analytic–synthetic and a priori–a posteriori distinctions. The discussion of Frege's account of these distinctions has usually focused on whether the conception of logic which informs his definition of analyticity undermines his claim to have shown that arithmetic is analytic in a sense that Kant would have been concerned to deny. I will argue that Frege has an elegant explanation of the *apriority* of arithmetic, one that challenges Kant even if Frege's claim to have reduced arithmetic to logic is rejected. In the course of discussing Frege's explanation of the apriority of arithmetic, I will also clarify certain fundamental differences between his and Whitehead and Russell's theories of number – differences which bear importantly on their respective accounts of the nature of arithmetical knowledge. I will show that even if Frege's understanding of Kant was defective in every detail, knowing what he took himself to be reacting against and correcting in Kant's philosophy of arithmetic is of interest for the light it casts on the development of logicism.

[1] Frege (1884). My citations of Frege (1879), (1884), and (1893/1903) – his *Begriffsschrift*, *Grundlagen*, and *Grundgesetze*, respectively – use the mnemonic abbreviations, *Bg*, *Gl*, and *Gg*, followed by the relevant page or section number.

1.1 FREGE'S INTEREST IN RIGOR

It may seem obvious that Frege's interest in rigor – his interest in providing a framework in which it would be possible to cast proofs in a canonical gap-free form – is driven by the problem of providing a proper justification for believing the truth of the propositions of arithmetic. One can easily find quotations from Frege which show that he sometimes at least wrote as if he took his task to be one of securing arithmetic; and it would be foolish to deny that the goals of cogency and consistency were an important part of the nineteenth-century mathematical tradition of which Frege was a part. Nevertheless, I think it can be questioned how far worries about the consistency or cogency of mathematics, generated perhaps by a certain incompleteness of its arguments, were motivating factors for Frege's logicism or for the other foundational investigations of the period.

There is another, largely neglected, component to Frege's concern with rigor that not only has an intrinsic interest, but also elucidates his views on the nature and significance of intuition in mathematical proof and his conception of his mathematical and foundational accomplishments. According to this component, Frege's concern with rigor is predominantly motivated by his desire to show that arithmetic does not depend on Kantian intuition, a concern Frege inherited from the tradition in analysis initiated by Cauchy and Bolzano, and carried forward by Weierstrass, Cantor, and Dedekind.

Shortly after the publication of *Begriffsschrift* Frege wrote a long study of its relationship to Boole's logical calculus.[2] The paper carries out a detailed proof (in the notation of *Begriffsschrift*) of the theorem that the sum of two multiples of a number is a multiple of that number. In addition to the laws and rules of inference of *Begriffsschrift*, Frege appeals only to the associativity of addition and to the fact that zero is a right identity with respect to addition. He avoids the use of mathematical induction by applying his definition of *following in a sequence* to the case of the number series. The paper also includes definitions of a number of elementary concepts of analysis (again in the notation of *Begriffsschrift*). It has been insufficiently emphasized that neither in this paper nor in *Begriffsschrift* does Frege suggest that the arithmetical theorems proved there are not correctly regarded as self-evident, or that without a *Begriffsschrift*-style proof, they and the propositions that depend on them might reasonably be doubted. Frege's point is rather that without gap-free proofs one might be misled into

[2] Frege (1880/1), published only posthumously.

thinking that arithmetical reasoning is based on intuition. As he puts the matter in the introductory paragraph to Part III of *Begriffsschrift*:

> Throughout the present [study] we see how pure thought, irrespective of any content given by the senses or even by an intuition *a priori*, can, solely from the content that results from its own constitution, bring forth judgements that at first sight appear to be possible only on the basis of some intuition.

The same point is made in *Grundlagen* when, near the end of the work (sects. 90–1), Frege comments on *Begriffsschrift*. Frege is quite clear that the difficulty with gaps or jumps in the usual proofs of arithmetical propositions is not that they might hide an unwarranted or possibly false inference, but that their presence obscures the true character of the reasoning:

> In proofs as we know them, progress is by jumps, which is why the variety of types of inference in mathematics appears to be so excessively rich; the correctness of such a transition is immediately self-evident to us; whereupon, since it does not obviously conform to any of the recognized types of logical inference, we are prepared to accept its self-evidence forthwith as intuitive, and the conclusion itself as a synthetic truth – and this even when obviously it holds good of much more than merely what can be intuited.
>
> On these lines what is synthetic and based on intuition cannot be sharply separated from what is analytic . . .
>
> To minimize these drawbacks, I invented my concept writing. It is designed to produce expressions which are shorter and easier to take in . . . so that no step is permitted which does not conform to the rules which are laid down once and for all. It is impossible, therefore, for any premiss to creep into a proof without being noticed. In this way I have, without borrowing any axiom from intuition, given a proof of a proposition[3] which might at first sight be taken for synthetic.

It might seem that to engage such passages it is necessary to enter into a detailed investigation of the Kantian concept of an a priori intuition. But to understand Frege's thought it is sufficient to recall that for the Kantian mathematical tradition of the period our a priori intuitions are of space and time, and the study of space and time falls within the provinces of geometry and kinematics. It follows that the dependence of a basic principle of arithmetic on some a priori intuition would imply that arithmetic lacks the autonomy and generality we associate with it. To establish its basic principles, we would have to appeal to our knowledge of space and time; and then arithmetical principles, like those expressing mathematical induction and various structural properties of the ancestral, would ultimately

[3] The proposition to which Frege refers is the last proposition proved in *Begriffsschrift* – Proposition 133 – which states that the ancestral of a many–one relation satisfies a restricted form of connectedness.