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Excerpt

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1. Sounding the solar cycle with helioseismology: Implications for asteroseismology

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1.1 Introduction

My brief for the IAC Winter School was to cover observational results on helioseismology, flagging where possible implications of those results for the asteroseismic study of solar-type stars. My desire to make such links meant that I concentrated largely on results for low angular-degree (low- l) solar p modes, in particular results derived from “Sun-as-a-star” observations (which are, of course, most instructive for the transfer of experience from helioseismology to asteroseismology). The lectures covered many aspects of helioseismology – modern helioseismology is a diverse field. In these notes, rather than discuss each aspect to a moderate level of detail, I have instead made the decision to concentrate on one theme, that of “sounding” the solar activity cycle with helioseismology. I cover the topics from the lectures and I also include some new material, relating both to the lecture topics and to other aspects I did not have time to cover. Implications for asteroseismology are developed and discussed throughout.

The availability of long time series data on solar-type stars, courtesy of the NASA Kepler Mission (Chaplin *et al.*, 2010; Gilliland *et al.*, 2010) and the French-led CoRoT satellite (Appourchaux *et al.*, 2008), is now making it possible to “sound” stellar cycles with asteroseismology. The prospects for such studies have been considered in some depth (Chaplin *et al.*, 2007a, 2008a; Metcalfe *et al.*, 2007; Karoff *et al.*, 2009, e.g.), and in the last year the first convincing results on stellar-cycle variations of the p-mode frequencies of a solar-type star (the F -type star HD49933) were reported by García *et al.* (2010). This result is important for two reasons: first, the obvious one of being the first such result, thereby demonstrating the feasibility of such studies; and second, the period of the stellar cycle was evidently significantly shorter than the 11-year period of the Sun (probably between 1 and 2 years). If other similar stars show similar short-length cycles, there is the prospect of being able to “sound” perhaps two or more complete cycles of such stars with Kepler (assuming the mission is extended, as expected, to 6.5 year or more). The results on HD49933 may be consistent with the paradigm that stars divide into two groups, activity-wise, with stars in each group displaying a similar number of rotation periods per cycle period (e.g., see Böhm-Vitense, 2007), meaning solar-type stars with short rotation periods – HD49933 has a surface rotation period of about 3 days – tend to have short cycle periods. We note that Metcalfe *et al.* (2010) recently found another F -type star with a short (1.6-year) cycle period (using chromospheric H & K data). Extension of the Kepler Mission will, of course, also open the possibility of detecting full swings in activity in stars with cycles having periods up to approximately the length of the solar cycle.

The rest of my notes break down as follows. Section 1.2 gives an introductory overview of the solar cycle, as seen in helioseismic data. A brief history of observations of solar cycle changes in low-angular degree (low- l) solar p modes is given in Section 1.2.1. Then, in Section 1.2.2, we consider the causes of the observed changes in the mode frequencies; and in Section 1.2.3, we discuss variations in the mode powers and damping rates, and what the relative sizes of those changes imply for the underlying cause.

Section 1.3 considers several subtle ways in which the stellar activity cycles can affect values of, and inferences made from, the mode parameters. Concepts are introduced using the example of the solar cycle, and the impact it has on low- l p modes observed in

Sun-as-a-star data. Implications for asteroseismic observations of solar-type stars are then developed. We start in Section 1.3.1 with a discussion of the impact of stellar activity on estimates of the mode frequencies. This is followed in Section 1.3.2 by a similar discussion for frequency separation ratios. Finally, Section 1.3.3 explains how mode peaks in the frequency-power spectrum can be “distorted” by stellar cycles, rendering commonly used fitting models inappropriate.

In Section 1.4 we develop a simple model to illustrate the impact on the mode frequencies of the range in latitudes covered by near-surface magnetic activity on solar-type stars, and discuss how the angle of inclination can affect significantly the observed frequency shifts (because that angle affects which mode components are visible in the observations). We also show how measurements of the frequency shifts of modes having different angular and azimuthal degrees may be used to make inferences on the spatial distribution of the near-surface magnetic activity on solar-type stars.

We end in Section 1.5 by thinking somewhat longer-term, and consider how much low- l data would be needed to measure evolutionary changes of the solar p-mode frequencies.

1.2 The seismic solar cycle: overview

A rich and diverse body of observational data is now available on temporal variations of the properties of the global solar p modes. The signatures of these variations are correlated strongly with the well-known 11-year cycle of surface activity. The search for temporal variations of the p-mode properties began in the early 1980s, following accumulation of several years of global seismic data. The first positive result was reported by Woodard and Noyes (1985), who found evidence in observations made by the Active Cavity Radiometer Irradiance Monitor (ACRIM) instrument, on board the Solar Maximum Mission (SMM) satellite, for a systematic decrease of the frequencies of low- l p modes between 1980 and 1984. The first year coincided with high levels of global surface activity, while during the latter period activity levels were much lower. The modes appeared to be responding to the Sun’s 11-year cycle of magnetic activity. Woodard and Noyes (1985) found that the frequencies of the most prominent modes had decreased by roughly 1 part in 10,000 between the activity maximum and minimum of the cycle. By the late 1980s, an in-depth study of frequency variations of global p modes, observed in the Big Bear data, had demonstrated that the agent of change was confined to the outer layers of the interior (Libbrecht and Woodard, 1990).

Accumulation of data from the new networks and instruments has made it possible to study the frequency variations to unprecedented levels of detail, and has revealed signatures of subtle, structural change in the subsurface layers. The discovery of solar-cycle variations in mode parameters associated with the excitation and damping (e.g., power, damping rate, and peak asymmetry) followed. Patterns of flow that penetrate a substantial fraction of the convection zone have also been uncovered as well as possibly signatures of changes in the rotation rate of the layers that straddle the tachocline, and much more recently evidence for a quasi-periodic 2-year signal, superimposed on the solar-cycle variations of the mode frequencies.

The modern seismic data give unprecedented precision on measurements of frequency shifts. Examples of average frequency shifts for low- l data are shown in Fig. 1.1, for Sun-as-a-star data collected by the ground-based Birmingham Solar-Oscillations Network (BiSON) and the Global Oscillations at Low Frequency (GOLF) instrument on board the ESA/NASA Solar and Heliospheric Observatory (SOHO). From observations of the medium- l frequency shifts it is possible to produce surface maps showing the strength of the solar-cycle shifts as a function of latitude and time (Howe *et al.*, 2002), like the example shown in Fig. 1.2 (which is made from Global Oscillations Network Group (GONG) data). These maps bear a striking resemblance to the butterfly diagrams that

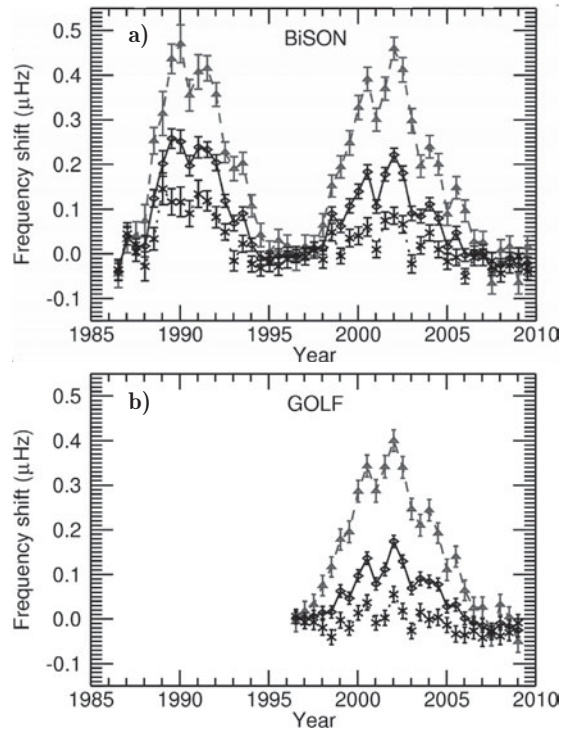


FIG. 1.1. Average frequency shifts of most prominent low- l solar p modes, as measured in: *a*) BiSON data and *b*) GOLF data over the last two 11-year activity cycles. The three curves show results for averages made over different ranges in frequency: 1,880 to 3,710 μHz (diamonds, joined by solid line); 1,880 to 2,770 μHz (crosses, joined by dotted line); 2,820 to 3,710 μHz (triangles, joined by dashed line). (From Fletcher *et al.*, 2010.)

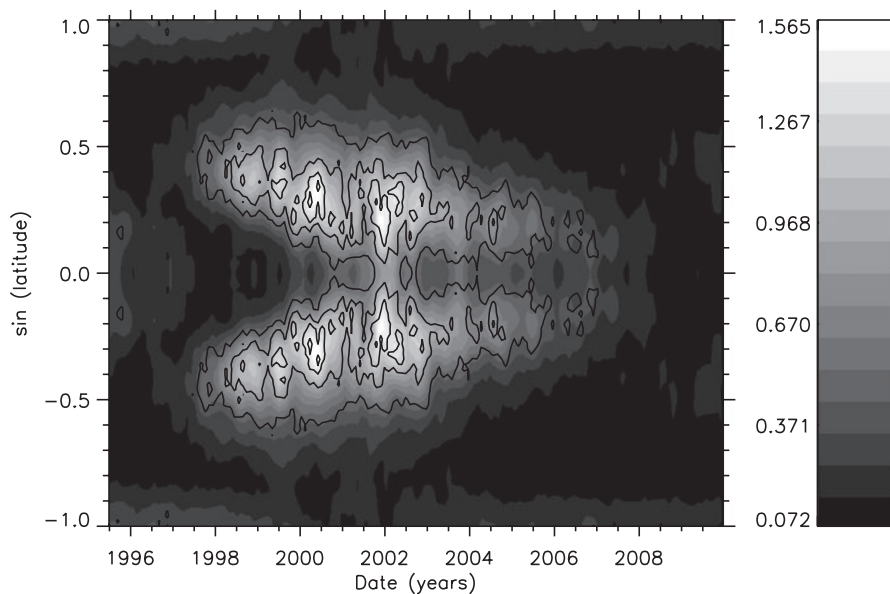


FIG. 1.2. Mode frequency shifts (in μHz) as a function of time and latitude. The values come from analysis of GONG data. The contour lines indicate the surface magnetic activity. (Figure courtesy of R. Howe.)

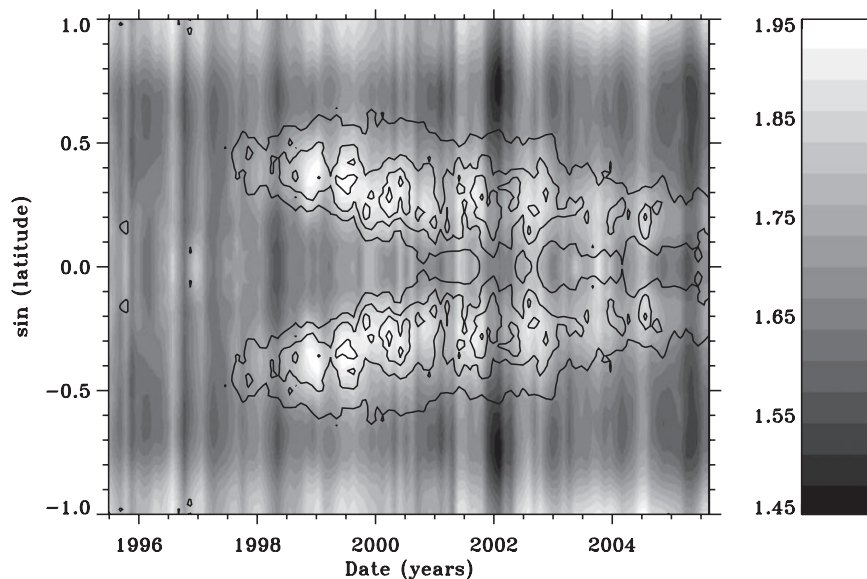


FIG. 1.3. Mode line width (in μHz) as a function of time and latitude. The values come from analysis of GONG data. The contour lines indicate the surface magnetic activity. (Courtesy of R. Howe.)

show spatial variations in the strength of the surface magnetic field over time. The implication is that the frequency shift of a given mode depends on the strength of that component of the surface magnetic field that has the same spherical harmonic projection on the surface. Similar maps may also be made for variations observed in the p-mode powers and damping rates (Komm *et al.*, 2002), which, like the frequency maps, show a close spatial and temporal correspondence with the evolution of active-region field (Fig. 1.3).

1.2.1 The seismic solar cycle at low angular degree

As noted above, the first evidence for activity-related changes to the low- l p modes of the Sun was reported by Woodard and Noyes (1985). These changes were soon confirmed (and the results extended) by Pallé *et al.* (1989) and Elsworth *et al.* (1990). Almost two decades on, there now exists an extensive literature devoted to studies of low- l mode parameter variations. Since the frequencies are by far the most precisely determined parameters, it is hardly surprising that analyses of the frequencies threw up the first positive results for solar-cycle changes. Evidence for changes in mode power followed next (Pallé *et al.*, 1990a; Anguera Gubau *et al.*, 1992; Elsworth *et al.*, 1994). The first tentative claims for changes in mode line width (damping) were made by Pallé *et al.* (1990b). Subsequently, Toutain and Wehrli (1997) and Appourchaux (1998) provided stronger evidence in support of an increase in damping with activity, and these claims were confirmed beyond all doubt by Chaplin *et al.* (2000). (Komm *et al.*, 2000, did likewise for medium l modes at about the same time.) Peak asymmetry is the most recent addition to the list of parameters that show solar-cycle variations (Jiménez-Reyes *et al.*, 2007). Careful measurement of variations in the powers, damping rates, and peak asymmetries – all parameters associated with the excitation and damping – are allowing studies to be made of the impact of the solar cycle on the convection properties in the near-surface layers.

Frequencies of modes below $\approx 4,000 \mu\text{Hz}$ are observed to increase with increasing activity. For the most prominent modes at around $3,000 \mu\text{Hz}$, the size of the shift is about $0.4 \mu\text{Hz}$ between activity minimum and maximum. Furthermore, higher-frequency modes

experience a larger shift than their lower-frequency counterparts. This frequency dependence suggests that the perturbations responsible for the frequency shifts are located very close to the solar surface. The upper turning points of the modes – which for low-degree modes are effectively independent of l – lie deeper in the Sun for low-frequency modes than they do for high-frequency modes: higher-frequency modes are as such more sensitive to surface perturbations.

That the shifts do not scale like ν_{nl}/L , where ν_{nl} is mode frequency and $L = \sqrt{l(l+1)}$, rules out the possibility that the perturbation is spread throughout a significant fraction of the solar interior. This may be understood by thinking classically, in terms of ray paths followed by the acoustic waves. When a wave reaches the lower turning point of its cavity it will by definition be moving horizontally, and its phase speed will be equal to

$$c = \frac{\omega_{nl}}{k} = \frac{2\pi\nu_{nl}R}{\sqrt{l(l+1)}} \propto \nu_{nl}/L, \quad (1.1)$$

where k is the horizontal wavenumber and R the outer cavity radius. The ratio ν_{nl}/L therefore maps to the location of the lower boundary of the cavity, and hence the cavity size. Since the shifts do not scale like this ratio, we conclude that the perturbation must be confined within a narrow layer, and not spread so widely that it covers the entirety of the cavities of many of the modes.

At frequencies above 4,000 μHz the size of the shift decreases, and also changes sign above $\simeq 4,500 \mu\text{Hz}$, meaning that these very high-frequency modes suffer a reduction in frequency as activity levels rise (Anguera Gubau *et al.*, 1992; Chaplin *et al.*, 1998; Jiménez-Reyes *et al.*, 2001; Gelly *et al.*, 2002; Salabert *et al.*, 2004).

Detailed comparison of the low- l frequency shifts with changes in various disc-averaged proxies of global surface activity provides further tangible input to the solar cycle studies. This is because different proxies show differing sensitivity to various components of the surface activity. While the changes in frequency are observed to correlate fairly well with contemporaneous changes in global proxies, the match is far from perfect. Jiménez-Reyes *et al.* (1998) were the first to show that the relation of the frequency shifts to variations in the proxies was markedly different on the rising and falling parts of the 11-year Schwabe cycle. Since the magnitudes of the shifts should reflect the different spatial sensitivities of modes of different angular and azimuthal degree to the time-dependent variation of the surface distribution of the activity, a better choice for the activity proxy would clearly be one that has been decomposed to have a similar spatial distribution as the mode under study. Chaplin *et al.* (2004a) and Jiménez-Reyes *et al.* (2004) have shown that the sizes of the low- l shifts do indeed scale better with activity proxies that have the same spherical harmonic projection as the modes.

Chaplin *et al.* (2007b) compared frequency changes in 30 years of BiSON data with variations in six well-known activity proxies. Interestingly, they found that only activity proxies having good sensitivity to the effects of weak-component magnetic flux – which is more widely distributed in latitude than the strong flux in the active regions – were able to follow the frequency shifts consistently over the three cycles.

The unusual behavior during the most recent solar minimum (straddling solar cycles 23 and 24) of many diagnostics and probes of solar activity has raised considerable interest and debate in the scientific community (e.g., see the summary by Sheeley, 2010). The minimum was unusually, and unexpectedly, extended and deep. Polar magnetic fields were very weak, and the open flux was diminished compared to other preceding minima.

Helioseismology has been used to probe the behavior of sub-surface flows during the minimum. Howe *et al.* (2009) found that the equatorward progression of the lower branches of the so-called torsional oscillations (east–west flows) was late in starting compared to previous cycles. They flagged this delayed migration as a possible precursor of the delayed onset of cycle 24. The meridional (north–south) flow also carries a signature of the solar cycle, which converges toward the active-region latitudes and also intensifies

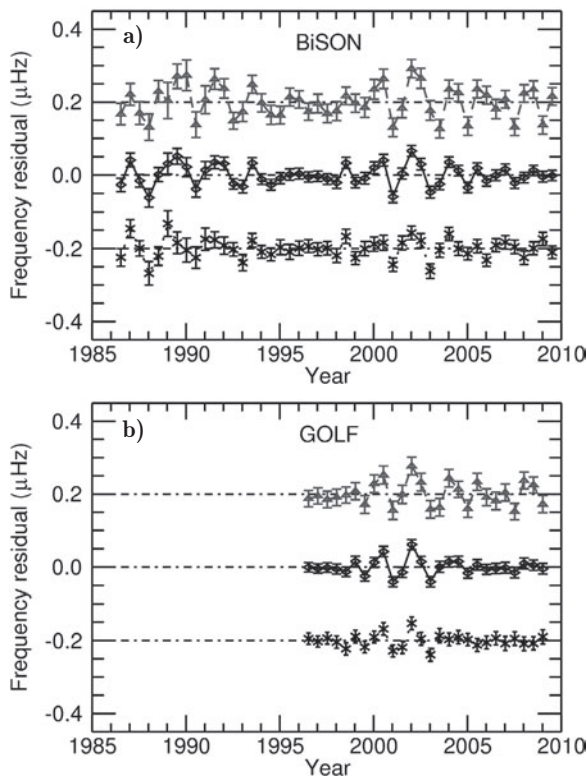


FIG. 1.4. Frequency residuals that remain after the long-term solar-cycle variation have been removed from the average frequency shifts of: *a*) BiSON data and *b*) GOLF data. The three curves in each panel show results for averages made over different ranges in frequency: 1,880 to 3,710 μHz (diamonds, joined by solid line); 1,880 to 2,770 μHz (crosses, joined by dotted line); 2,820 to 3,710 μHz (triangles, joined by dashed line). (Plot from Fletcher *et al.*, 2010.)

in strength as activity increases: González Hernández *et al.* (2010) found that during the current minimum this component had developed to detectable levels even before the visual onset of magnetic activity on the solar surface.

The globally coherent acoustic properties of the recent solar minimum have been studied extensively with low-degree p modes (e.g., Broomhall *et al.*, 2009; Salabert *et al.*, 2009) and medium-degree p modes (Tripathy *et al.*, 2010). These studies have shown that while the surface proxies of activity (e.g., the 10.7-cm radio flux) were quiescent and very stable during the minimum, the p-mode frequencies showed much more variability. Tripathy *et al.* (2010) noted further surprising behavior compared to the previous cycle 22–23 minimum, that is, an apparent anticorrelation of the p-mode frequency shifts and the surface proxies of activity.

Broomhall *et al.* (2009) had suggested the possible presence of a quasi-biennial modulation of the frequencies of the low-degree modes, superimposed upon the well-established ~ 11 -year variation of the frequencies. This has since been confirmed by further in-depth analysis, which reveals a signature that is consistent in the frequencies extracted from BiSON and GOLF data (Fletcher *et al.*, 2010), as shown in Fig. 1.4. This plots the frequency residuals that remain after the long-term solar-cycle variation has been removed from the average frequency shifts. The residuals show significant variability, with a period of around 2 years, that variability being more pronounced at times of higher surface activity. The fact that this biennial signature has similar amplitude in the low-frequency *and* the high-frequency modes used in this analysis suggests that its origins lie deeper than the very superficial layers responsible for the 11-year shifts.

Finally in this section on the frequencies, we note that from appropriate combinations of low- l frequencies Verner *et al.* (2006) uncovered apparent solar-cycle variations in the amplitude of the depression in the adiabatic index, Γ_1 , in the He II zone. These variations presumably reflect the impact of the changing activity on the equation of state of the gas in the layer, and confirmed the findings of Basu and Mandel (2004), who used the more numerous data available from medium- l frequencies.

1.2.2 *What is the cause of the frequency shifts?*

Broadly speaking, the magnetic fields can affect the modes in two ways. They can do so directly, by the action of the Lorentz force on the gas. This provides an additional restoring force, the result being an increase of frequency, and the appearance of new modes. Magnetic fields can also influence matters indirectly, by affecting the physical properties in the mode cavities and, as a result, the propagation of the acoustic waves within them. This indirect effect can act both ways, to either increase or decrease the frequencies.

We begin this section with a back-of-the-envelope calculation, which makes an (admittedly) approximate prediction of the impact on the frequency of a typical low- l mode of changes in stratification brought about by the near-surface magnetic field in sunspots. The calculation is instructive, in that the size of the predicted frequency shift it gives is broadly in line with the observations, providing further evidence in support of the shifts being caused by near-surface perturbations due to the presence of magnetic field.

As we have seen, frequencies of the most prominent modes increase with increasing magnetic activity. Here, we consider the indirect effect of the near-surface magnetic field on the near-surface properties, and hence the frequencies of these modes. If magnetic fields modify the surface properties in such a way as to reduce the effective size of the mode cavity, the required increase of frequency will result.

In regions of strong magnetic field – such as those occupied by sunspots – there will be a gas pressure deficit (assuming those regions to be in pressure equilibrium with their field-free surroundings). This is because gas and magnetic pressure combine within the magnetic regions, while only gas pressure acts in the field-free regions. Sunspots are also characterized by a reduced temperature relative to the surroundings. The central part of the spot therefore exhibits a lower pressure, temperature, and density than the surroundings, resulting in the so-called Wilson Depression. Values of pressure, temperature, and density found at the surface in the field-free plasma are only reached at some depth beneath the surface in strong-field regions. Recent measurements suggest that for a sunspot the typical size of this depression is about 1,000 km (Watson *et al.*, 2009).

Let us assume that 1,000 km corresponds approximately to the amount, δR , by which the mode cavities are reduced in size beneath sunspots. The fractional area of the solar surface occupied by sunspots reaches $\sim 0.5\%$ at modern cycle maxima. We therefore obtain a net, surface-averaged estimate of δR via

$$\langle \delta R \rangle \approx 0.5\% \times 1000 \text{ km} \approx 5 \text{ km}. \quad (1.2)$$

The sound speed, c , at the solar surface is approximately 10 km s^{-1} , implying a reduction in the travel time in the mode cavity, due to the “shrinkage” at the surface, of

$$\delta T \approx 5 \text{ km} / 10 \text{ km s}^{-1} \approx 0.5 \text{ s}. \quad (1.3)$$

The travel time across the cavity, T , is related to the large frequency separation, $\Delta\nu$, via:

$$\Delta\nu \simeq \left(2 \int_0^R \frac{dr}{c} \right)^{-1} \simeq (2T)^{-1}. \quad (1.4)$$

For the Sun, $\Delta\nu = 135 \mu\text{Hz}$, implying $T \approx 3700$ s. The fractional change (reduction) in T is therefore:

$$\delta T/T \simeq 0.5/3700 \simeq 1.4 \times 10^{-4}. \quad (1.5)$$

Since $\delta\nu/\nu = -\delta T/T$, the predicted increase in frequency of a mode at $\approx 3,000 \mu\text{Hz}$ will be

$$\delta\nu = 3000 \times 1.4 \times 10^{-4} \simeq 0.4 \mu\text{Hz}. \quad (1.6)$$

This estimate is of a very similar size to the observed frequency shifts.

What does detailed modelling suggest? Perhaps the most significant contribution of recent years in this area is that of Dziembowski and Goode (2005). Their results suggest that the indirect effects dominate the perturbations, and that the magnetic fields are too weak in the near-surface layers for the direct effect to contribute significantly to the observed frequency shifts. However, Dziembowski and Goode (2005) also found some evidence to suggest that the direct effect may play a more important rôle for low-frequency modes, at depths beneath the surface where the magnetic field is strong enough to give a significant direct contribution to the frequency shifts. We now go on to explain how dependence of the frequency shifts on mode inertia and mode frequency can tell us something about the location and nature of the perturbations.

We begin by noting that when the frequency shifts are multiplied by the mode inertia, and then normalized by the inertia of a radial mode of the same frequency, the modified shifts are found to be a function of frequency alone. This in effect removes any l dependence of the shifts (at fixed frequency, the higher the l , the larger is the observed frequency shift). The observed l dependence may be understood in terms of, for example, a physical interpretation of the mode inertia. The normalized inertia, I_{nl} , may be defined according to (Christensen-Dalsgaard and Berthomieu, 1991):

$$I_{nl} = M_{\odot}^{-1} \int_V |\xi|^2 \rho dV = 4\pi M_{\odot}^{-1} \int_0^R |\xi|^2 \rho r^2 dr = M_{nl}/M_{\odot} \quad (1.7)$$

where ξ is the (surface-normalized) displacement associated with the mode, and the integration is performed over the volume V of the Sun, which has mass M_{\odot} . The mode mass M_{nl} is therefore the interior mass affected by the perturbations associated with the mode. As l increases, so M_{nl} decreases, and the more sensitive a mode will be to a near-surface perturbation of a given size (giving a larger frequency shift). One may therefore render the shifts l independent by multiplying them by the inertia ratio Q_{nl} (Christensen-Dalsgaard and Berthomieu, 1991), which is given by:

$$Q_{nl} = I_{nl}/\bar{I}(\nu_{nl}). \quad (1.8)$$

Multiplication of the raw shifts by Q_{nl} is indeed seen to collapse the shifts of different l onto a single curve. As shown in Fig. 1.5, this then allows one to combine data spanning a range in l , which reduces errors, giving tighter constraints on frequency dependence of the shifts.

Chaplin *et al.* (2001) studied in detail the frequency dependence of the inertia-ratio-corrected shifts, $\delta\nu_{nl}Q_{nl}$, of both low- l modes and medium- l modes up to $l = 150$. They fitted these data to a power law of the form

$$\delta\nu_{nl}Q_{nl} = \frac{c}{I_{nl}} \nu_{nl}^{\alpha}, \quad (1.9)$$

where the power-law index is α and c is a constant. They found that $\alpha \simeq 2$ for $\nu \geq 2,500 \mu\text{Hz}$, while α is approximately zero for $\nu < 2,500 \mu\text{Hz}$. Rabello-Soares *et al.* (2008) repeated the exercise for medium- l and high- l modes. They extracted very similar behavior, and, because they had more medium- and high- l data at low frequencies, they

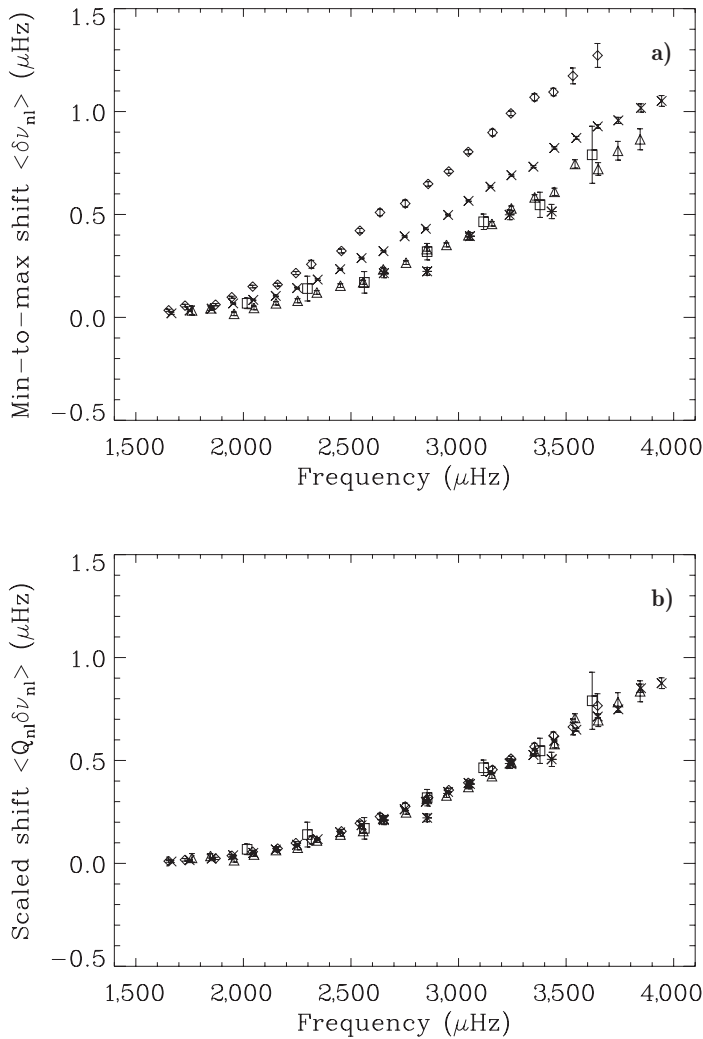


FIG. 1.5. *a)* Frequency shifts, $\delta\nu$, plotted as a function of mode frequency, for results from different instruments over different ranges in l (diamonds: GONG, $l = 100$ to 150 ; crosses: GONG, $l = 30$ to 90 ; triangles: GONG, $l = 4$ to 25 ; squares: BiSON, $l = 0$ to 2 ; asterisks: LOI, $l = 0$ to 8). *b)* Scaled frequency shifts, after multiplication by the inertia ratio Q_{nl} . (From Chaplin *et al.*, 2001.)

were able to extend the analysis to $\nu < 2,000 \mu\text{Hz}$ where they found that α appears to change sign and go negative. The high- l modes provide important information, as they are confined in the layers close to the surface where the physical changes responsible for the frequency shifts are also located.

What do the values of α imply? If the perturbation is located within the photosphere (but is confined to extend over an extent no longer than one pressure scale height), then we might expect $\alpha \simeq 3$ (e.g., Libbrecht and Woodard, 1990). If instead the perturbation extends beneath the surface, the frequency dependence will be weaker and α will get smaller (Gough, 1990). The observed values of α in the abovementioned frequency regimes are both less than 3, suggesting that the perturbations cannot arise solely in the photosphere. Since α is smaller for the lower-frequency modes, this suggests that the perturbation extends to greater depths, the lower in frequency one goes. This is consistent with the inferences made by Dziembowski and Goode (2005).

1.2.3 *Solar-cycle variations of mode power and mode damping*

Variations in mode damping have now been uncovered in all the major low- l datasets (see Chaplin, 2004, and references therein). The trend found is an increase of about 20 percent between activity minimum and maximum, with some suggestion of the variations being peaked in size at about $\simeq 3,000 \mu\text{Hz}$. Mode powers are at the same time observed to decrease by about the same fractional amount, while the mode heights decrease by twice the amount. Use of the analogy of a damped, randomly forced oscillator is particularly instructive for understanding these results.

We define our oscillator according to the equation:

$$\ddot{x}(t) + 2\eta\dot{x}(t) + (2\pi\nu_0)^2x(t) = K\delta(t - t_0), \quad (1.10)$$

where $x(t)$ is the displacement, ν_0 the natural frequency of the oscillator, η the linear damping constant, and K is the amplitude of the forcing function (assumed to be a random Gaussian variable), with “kicks” applied at times t_0 , $\delta(t - t_0)$ being the delta function.

Provided that $\mathcal{F}(\nu)$ – the frequency spectrum of the forcing function – is a slowly varying function of ν , and $\eta \ll 2\pi\nu_0$, the power spectral density (PSD) in the frequency domain will be a Lorentzian profile, that is,

$$\text{PSD}(\nu) \propto \mathcal{F}(\nu) \left(1 + \left(\frac{\nu - \nu_0}{\eta/2\pi} \right)^2 \right)^{-1}. \quad (1.11)$$

This holds for both the spectrum of the displacement, $x(t)$, and the velocity, $\dot{x}(t)$. The FWHM of the peak in cyclic frequency is

$$\Delta = \frac{\eta}{\pi}. \quad (1.12)$$

The maximum power spectral density, or height, of the resonant peak is:

$$H \propto \frac{\mathcal{F}(\nu)}{\eta^2}. \quad (1.13)$$

The total mean-square power (variance in the time domain) is proportional to peak height times width, that is, $P \propto H\Delta$, so that

$$P \propto \frac{\mathcal{F}(\nu)}{\eta}. \quad (1.14)$$

The energy (kinetic plus potential) of a resonant mode with associated inertia I is given by:

$$E = MP. \quad (1.15)$$

The rate at which energy is supplied to (and dissipated by) the modes, dE/dt , is readily derived by again having recourse to the oscillator analogy. The amplitude of the oscillator is attenuated in time by the factor $\exp(-\eta t)$, and its energy is proportional to the amplitude squared. Hence, we may write

$$E = (\text{constant}) \times \exp(-2\eta t). \quad (1.16)$$

It follows that:

$$\log E = -2\eta t + \log(\text{constant}), \quad (1.17)$$

and taking derivatives:

$$dE/dt = -2\eta E. \quad (1.18)$$

If we combine Equations 1.14, 1.15, and 1.18, we have:

$$dE/dt = \dot{E} \propto -\mathcal{F}(\nu)I. \quad (1.19)$$