8. One-Loop Diagrams in the Bosonic String Theory

Our discussions of string scattering amplitudes in the first volume of this book were limited to tree diagrams. These are the lowest-order approximations to string scattering amplitudes. In principle, quantum corrections to the tree level or classical results should be obtained by a perturbation expansion derived from string quantum field theory. Our present state of knowledge does not make this possible. Historically, loop diagrams were constructed by using unitarity to construct loop diagrams from tree diagrams. This unitarization of the tree diagrams led, in time, to the topological expansion, as sketched in chapter 1.

As has been explained in chapters 1 and 7, the tree amplitudes for on-mass-shell string states can be represented by functional integrals over Riemann surfaces that are topologically equivalent to a disk (for open strings) or a sphere (for closed strings). Higher-order corrections are identified with functional integrals over surfaces of higher genus. An important ingredient in the calculation of scattering amplitudes is the correlation function of vertex operators corresponding to the external particles emitted from the surface. The possible world-sheet topologies include surfaces with holes or 'windows' cut out (for type I theories, where the surfaces have boundaries) or 'handles' attached. For theories with oriented strings the surfaces must be orientable. Similarly, for theories containing only closed strings the surfaces must be closed.

As the genus of a surface increases, the power of the coupling constant that accompanies it also increases. For example, adding a handle to a surface is equivalent to adding a loop of closed strings (as in fig. 8.1) and increases the order of a diagram by a factor of κ^2, where κ is the gravitational coupling constant. Cutting a window out of a surface (which is only possible in theories that contain open strings) adds a boundary and hence it increases the number of internal open strings (fig. 8.2a). The order of the diagram is increased by $g^2 \sim \kappa$ for each window, where g is the Yang–Mills coupling constant. However, the presence of a window does not always correspond to adding a loop of open strings. For example,

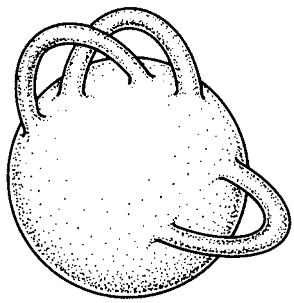
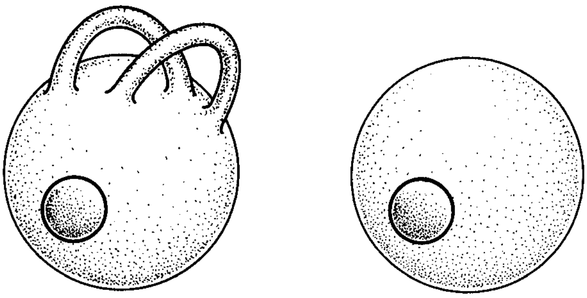


Figure 8.1. A handle added to a world sheet of arbitrary topology.



(*a*) (*b*)

Figure 8.2. Cutting a window out of a world sheet adds a boundary. This increases the number of internal open-string propagators as seen in (*a*). Cutting a window out of a spherical world sheet results in a diagram that is topologically a disk, as shown in (*b*).

cutting a window out of a sphere is a modification of the (type I) closed-string tree amplitude, which gives a world sheet that is topologically a disk with external closed-string particles attached at interior points of the surface (fig. 8.2*b*). Type I superstring theory is based on unoriented open and closed strings and therefore also includes nonorientable surfaces.

This topological classification of diagrams in string theories is certainly strikingly different from the classification of Feynman diagrams in point-particle field theory. In string theories there are far fewer diagrams to consider at each order in perturbation theory, and there is no meaningful separation of diagrams into tadpoles, mass insertions, vertex corrections, etc. At the one-loop level, the analysis of world-sheet path integrals is tractable. In fact, one-loop diagrams can be generated by the same operator methods that we used for tree diagrams in chapter 7. Beyond the one-loop level, the analysis of world-sheet path integrals involves somewhat esoteric mathematics, which we will not explore in this book.

In the bosonic theory calculations based on the covariant operator formalism require the same mathematical manipulations as those that arise in light-cone gauge, at least when the external on-shell states are taken to have vanishing + components of momentum. Given Lorentz invariance, amplitudes for external particles with momenta restricted in this way completely determine the amplitudes provided that there are not too many external states. Although we use the covariant method in most of this chapter, very similar techniques also apply to the light-cone gauge method in this special frame.

8.1 Open-String One-Loop Amplitudes

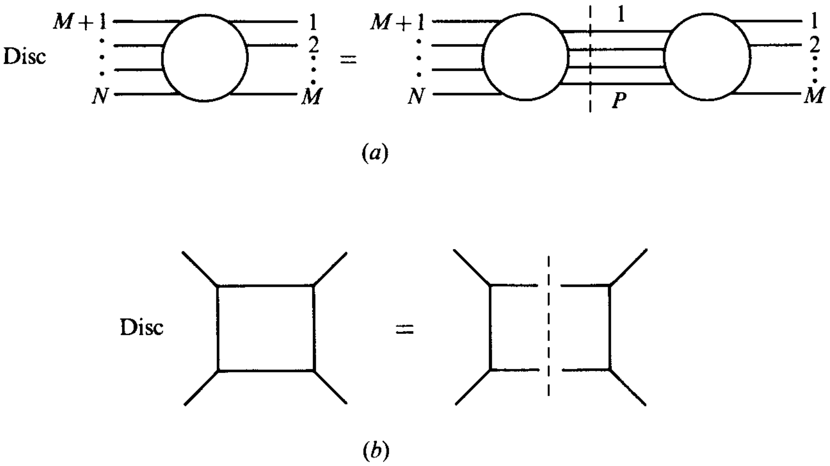


Figure 8.3. (a) Unitarity equates the discontinuity of a scattering amplitude (with M incoming and N outgoing particles) across a threshold cut (due to P intermediate particles) to the product of $M \to P$ and $P \to N$ scattering amplitudes integrated over intermediate state phase space. (b) At one loop, unitarity relates the discontinuity of a loop diagram to the integral of the product of two tree diagrams over the phase space for the intermediate on-shell two-particle states.

In point-particle theories the one-loop diagrams can be determined by unitarity in terms of tree diagrams without using the apparatus of second-quantized field theory. Unitarity requires that scattering amplitudes should have suitable branch cuts as a function of the Lorentz-invariant quantities formed out of the external momenta. These cuts arise from the regions of momentum space in which intermediate states are on their mass shells. For example, fig. 8.3*a* depicts the unitarity equation for an amplitude with M incoming and N outgoing particles. A given set of P intermediate on-shell physical states contributes to the discontinuity across the branch cut an amount that is proportional to the product of the amplitude for $M \to P$ particles multiplied by the amplitude for $P \to N$ particles integrated over the accessible phase space for the intermediate particles.

When expanded as a power series in the coupling constant this nonlinear equation relates the discontinuity of a one-loop amplitude to the product of two tree amplitudes. In this case, illustrated in fig. 8.3*b*, the number of intermediate states, P, is two. In particular, the form of the one-loop amplitude, including its normalization, is determined in terms of the tree diagrams up to an arbitrary entire function of these invariants. In the case of ordinary field theory, the arbitrary entire function corresponds to the arbitrariness associated with the renormalization procedure. In gauge invariant field theories, it is also necessary to avoid including in loop diagrams the contributions of timelike or longitudinally polarized gauge mesons. These contributions can be removed by going to a light cone or unitary gauge, or can be canceled by correctly including the Faddeev-Popov ghosts.

Similar considerations apply to the construction of the one-loop amplitudes in string theories from the tree diagrams. In this case the requirement of Regge behavior at high energies eliminates the ambiguity that exists in field theory. Regge behavior forces amplitudes to vanish in certain asymptotic regimes; addition of an entire function of the momenta to one-loop diagrams would inevitably spoil this property.

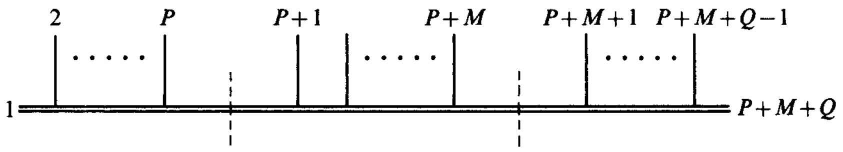


Figure 8.4. A general tree diagram with $P + M + Q$ ground-state particles factorized to give a tree with two arbitrary excited states and M ground states.

For example, the tree diagram of fig. 8.4 illustrates the interaction of $P + M + Q$ on-shell open-string states. It can be factorized as shown in the figure to obtain the amplitude for an arbitrary pair of 'excited' states to couple to M on-shell states. Ignoring the presence of unphysical states for the moment the one-loop amplitude is obtained by *sewing* the excited states together, *i.e.*, by inserting a propagator between the initial and the final excited states and summing over all possible states as well as integrating over their momenta. In the complete amplitude it is necessary to sum over loop diagrams with twists inserted in all possible ways in the internal propagators of the loop.

Just as in ordinary field theory, covariant string-theory formulas describe states of unphysical polarization circling in the loop. Care must be taken to somehow suppress their contribution. In early calculations of string loop diagrams, the propagation of unphysical states was avoided by inserting a rather complicated physical-state projection operator in the propagators. This ensured that the circulating particles corresponded only to physical states; the procedure was analogous to some early approaches to Yang–Mills theory. A more modern approach incorporates the Faddeev–Popov ghost modes in the calculations instead. This approach is far simpler, and is the approach that we will use in performing covariant calculations.

In the bosonic theory the inclusion of ghost modes is quite easy. The vertex operators, such as the tachyon vertex operator $e^{ik\cdot X}$, are constructed from X^μ only, without ghosts, where $X^\mu(\sigma,\tau)$ is the string coordinate defined in chapter 2. When ghosts are included in the formalism, these vertex operators are understood to include a unit operator in the ghost sector of the Fock space. The ghosts circulating around the loops can then cancel the contributions of unphysical states. This is their only role.

How can we be certain that the ghosts are really correctly canceling the contributions of the unphysical states? It is particularly important to address this question, since – pending a completely satisfactory derivation of loop amplitudes from a logically satisfying starting point – there is an element of guesswork in formulating the Feynman rules including the ghosts. To gain some insight into this important question, it is possible to do the calculations in light-cone gauge. In this case, there are no unphysical states propagating in the loop – neither states that violate the Virasoro conditions, nor null states, nor ghosts. All the states in the light-cone Fock space correspond to physical propagating degrees of freedom. The light-cone amplitudes are thus manifestly unitary – or at any rate, singularities that appear are due to physical intermediate states. It will be rather clear in our discussion that – at least for processes that are easily discussed in both formalisms – the light-cone approach gives the same answers that one obtains in the covariant treatment with ghosts. Ultimately, the rules involving Faddeev–Popov modes should be derived from a logically sound starting point, perhaps a gauge-invariant nonlinear field theory of strings.

A curious feature of string theories is that new singularities can arise due to divergences of sums over intermediate states. This feature already appeared in tree amplitudes, where we saw in chapters 1 and 7 that t-channel poles arise due to divergences in the sum over s-channel poles. In the case of loop diagrams even more remarkable things can happen. For instance, an open-string loop with suitable twists can actually give rise to closed-string poles. It was by trying to reconcile these singularities with unitarity that the significance of the critical dimension first became apparent; in the critical dimension, these singularities correspond to graviton poles, and (as we discussed in §1.5.6), they are the reason that a consistent string theory without gravity does not seem possible, at the quantum level.

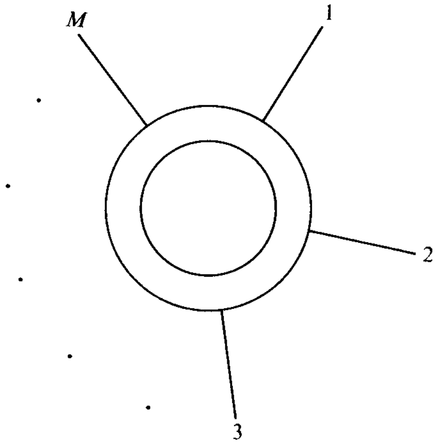


Figure 8.5. The planar loop diagram with M ground-state particles

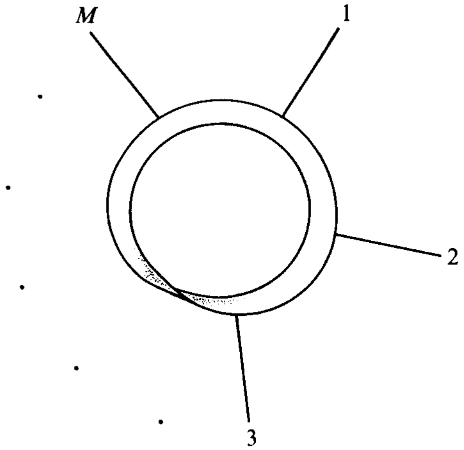


Figure 8.6. A nonorientable one-loop diagram with M external particles has a world sheet that is a Möbius strip.

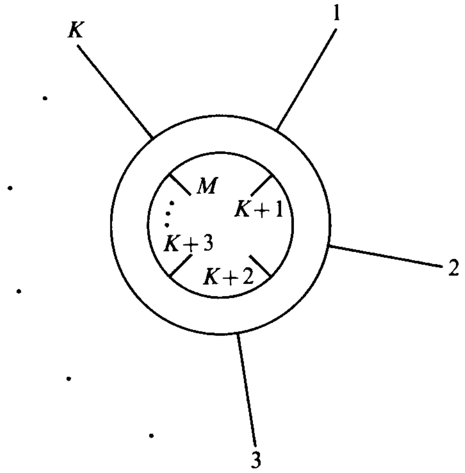


Figure 8.7. A nonplanar one-loop diagram in which K particles are attached to one boundary and $M - K$ to the other.

The simplest one-loop diagram in a theory of open strings corresponds to a process for which the world sheet is topologically an annulus or cylinder with M external states attached to one boundary as illustrated in fig. 8.5. (A world sheet with this topology is referred to as a *planar* diagram). The precise meaning of the parameters describing the annulus and the positions of the attached particles in this figure is explained later in this chapter. By including an odd number of twists in the world sheet it is possible to construct other one-loop diagrams associated with world sheets that are *nonorientable*, *i.e.*, Möbius strips having only one boundary (as in fig. 8.6). By using an even number of twists one can describe oriented surfaces in which particles are attached to both boundaries of the annulus as in fig. 8.7. These are called *nonplanar* diagrams. These various different contributions to the full one-loop open-string amplitude must be calculated separately, although much of the computation is similar for each of the diagrams. (In this respect, theories of oriented closed strings, which have only one diagram at each order, are a lot simpler.)

8.1.1 The Planar Diagrams

Let us consider bosonic open strings carrying group-theory quantum numbers of the type described in §6.1. Let n be the dimension of the fundamental representation of the gauge group – the representation of the charges that sit at the ends of the open string. Then the group-theory factor associated with the planar diagram (fig. 8.5) is

$$G_P = n\mathrm{tr}(\lambda_1\lambda_2\dots\lambda_M), \tag{8.1.1}$$

where the factor of n arises from the trace of the $n\times n$ unit matrix associated with the free boundary of the annulus. As in the case of tree diagrams, the matrices λ_r must be $n\times n$ matrices belong to the fundamental representation of the algebra of any of the allowed groups (*i.e.*, the classical groups $SO(n)$, $USp(n)$ and $U(n)$) if the states are at even mass levels. Hermitian matrices (denoted μ in §6.1) would be used for odd levels.

For simplicity and explicitness, we mostly consider processes in which the external states are either tachyons (an odd level) or massless vector particles (an even level), although essentially the same techniques can be used for arbitrary excited states. In either case the vertex for emitting an on-shell particle with momentum k_r at 'time' τ is denoted by $V(k_r,\tau)$, where

$$V(k_r,\tau) = e^{i\tau L_0}V(k_r,0)e^{-i\tau L_0}. \tag{8.1.2}$$

As in chapter 7, we frequently work with $x=e^{i\tau}$ and take x to be real, corresponding to a Wick-rotated time coordinate. In this case we write

$$V(k_r,x) = x^{L_0}V(k_r,1)x^{-L_0}. \tag{8.1.3}$$

We recall that, apart from the vertex (8.1.3), the main ingredient in the construction of tree diagrams in chapter 7 was the propagator, which for bosonic open strings was

$$\Delta = (L_0-1)^{-1}. \tag{8.1.4}$$

To associate an amplitude with the diagram of fig. 8.5, we include a vertex (8.1.2) for each external line, and a propagator (8.1.4) for each internal line. The closed loop is represented by a trace in the Fock space of the internal lines. Putting things together in this way, the amplitude that we define is

$$\begin{aligned} A_P(1,2,\dots,M) = g^M G_P \int d^Dp\, \mathrm{Tr}\big(&\Delta V(k_1,1)\Delta V(k_2,1) \\ &\dots\Delta V(k_M,1)\big). \end{aligned} \tag{8.1.5}$$

In the covariant formalism used here the trace runs over the infinite set of bosonic oscillator modes α_n^μ, $\mu=0,\dots,25$, as well as the ghost oscillators

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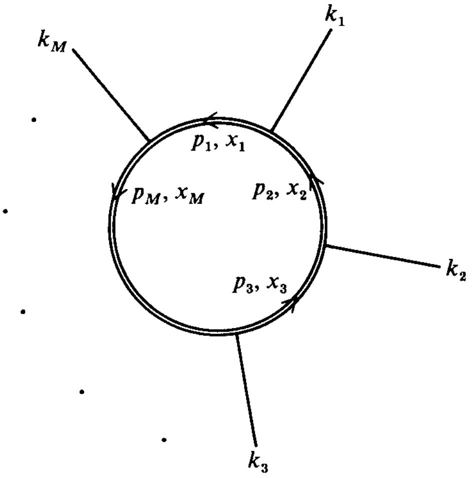

Figure 8.8. The kinematics for the calculation of the planar loop diagram.

b_n and c_n. In the light-cone gauge, the only modes entering in the trace would be the transverse oscillators α_n^i, $i = 1, \ldots, 24$. Just as in ordinary field theory, the poles of the propagators give rise to cuts associated with on-shell intermediate states.

The sequence of emitted particles $(1, 2, \ldots, M)$ corresponds to the order in which they are attached to the boundary in fig. 8.5. The cyclic property of the trace ensures that only the cyclic ordering matters. The full one-loop planar amplitude includes a sum over all cyclically inequivalent permutations of the external particles, each weighted with its own group-theory factor. The kinematics for this process is illustrated in fig. 8.8.

As in chapter 7, it is convenient to use the integral representation

$$\Delta = (L_0 - 1)^{-1} = \int_0^1 x^{L_0 - 2} dx \tag{8.1.6}$$

for the open-string propagator. The vertex operator for emitting an on-shell tachyon of momentum k^μ (with $k^2 = 2$) is given by

$$V_0(k, 1) =: e^{ik \cdot X(1)} : . \tag{8.1.7}$$

If the emitted particle is a massless vector boson, the vertex operator is given by

$$V(\zeta, k, 1) = \zeta \cdot \dot{X}(1) e^{ik \cdot X(1)}, \tag{8.1.8}$$

where ζ^μ is the polarization vector of the particle and $\zeta \cdot k = k^2 = 0$).

As explained in chapter 7, a convenient way to evaluate amplitudes with external vector particles is to use the vertex operator

$$V(k,\zeta,1) = \exp\{\zeta\cdot\dot X(1) + ik\cdot X(1)\}, \tag{8.1.9}$$

with the understanding that only the terms linear in the ζ's are to be kept. This vertex factorizes into a product of terms for each oscillator mode, which is helpful in the evaluation of the traces. (This will also be useful in the discussion of the heterotic string in the next chapter.) This vertex does not need to be normal ordered, since the only ordering factor that arises is an exponential involving ζ^2, which does not contribute to the terms linear in ζ^μ.

Let us now consider the one-loop planar diagram with M external tachyons. Inserting the integral representation for the propagators in (8.1.5) and using the fact that $x^{L_0}V(k_r,1) = V(k_r,x)x^{L_0}$, the expression for the loop can be written as

$$\begin{aligned} A_P(1,2,\ldots,M) &= g^M G_P \int_0^1 \prod_{i=1}^M dx_i \int d^Dp\,\mathrm{Tr}[V_0(k_1,x_1) \\ &\qquad \times V_0(k_2,x_1x_2)\ldots V_0(k_M,x_1\ldots x_M)w^{L_0-2}] \\ &= g^M G_P \int_0^1 \frac{dw}{w^2}\int_0^1 \prod_{r=1}^{M-1}\frac{d\rho_r}{\rho_r}\Theta(\rho_r-\rho_{r+1})I(1,\ldots,M), \end{aligned} \tag{8.1.10}$$

where

$$I(1,\ldots,M) = \int d^Dp\,\mathrm{Tr}\left(V_0(k_1,\rho_1)\ldots V_0(k_M,\rho_M)w^{L_0}\right) \tag{8.1.11}$$

and

$$\rho_r = x_1\ldots x_r, \tag{8.1.12}$$

$$w \equiv \rho_M = x_1\ldots x_M. \tag{8.1.13}$$

In writing (8.1.10), we have used the fact that the Jacobian for the transformation from the x_r variables that parametrize the individual propagators to the ρ_r variables is given by

$$\prod_{r=1}^M dx_r = dw\prod_{r=1}^{M-1}\frac{d\rho_r}{\rho_r}. \tag{8.1.14}$$

The variables ρ_r are integrated on the interval $(w,1)$ of the real axis of the complex ρ plane.