An Introduction to Continuum Mechanics, Second Edition

This best-selling textbook presents the concepts of continuum mechanics in a simple yet rigorous manner. The book introduces the invariant form as well as the component form of the basic equations and their applications to problems in elasticity, fluid mechanics, and heat transfer and offers a brief introduction to linear viscoelasticity. The book is ideal for advanced undergraduates and beginning graduate students looking to gain a strong background in the basic principles common to all major engineering fields and for those who will pursue further work in fluid dynamics, elasticity, plates and shells, viscoelasticity, plasticity, and interdisciplinary areas such as geomechanics, biomechanics, mechanobiology, and nanoscience. The book features derivations of the basic equations of mechanics in invariant (vector and tensor) form and specification of the governing equations to various coordinate systems, and numerous illustrative examples, chapter summaries, and exercise problems. This second edition includes additional explanations, examples, and problems.

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An Introduction to Continuum Mechanics, Second Edition

J. N. REDDY

Texas A & M University
To

Rohan, Asha, and Mira

Who have filled my life with joy
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List of Symbols

The symbols that are used throughout the book for various important quantities are defined in the following list. In some cases, the same symbol has different meaning in different parts of the book; it should be clear from the context.

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<th>Symbol</th>
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<td>$a$</td>
<td>Acceleration vector, $\frac{Dv}{Dt}$</td>
</tr>
<tr>
<td>$A$</td>
<td>Matrix of normalized eigenvectors [see Eq. (3.9.8)]</td>
</tr>
<tr>
<td>$B$</td>
<td>Left Cauchy–Green deformation tensor (or Finger tensor), $B = F \cdot F^T$; magnetic flux density vector</td>
</tr>
<tr>
<td>$\hat{B}$</td>
<td>Cauchy strain tensor, $\hat{B} = F^{-T} \cdot F^{-1}$; $\hat{B}^{-1} = B$</td>
</tr>
<tr>
<td>$B(,)$</td>
<td>Bilinear form</td>
</tr>
<tr>
<td>$c$</td>
<td>Specific heat, moisture concentration</td>
</tr>
<tr>
<td>$c_v, c_p$</td>
<td>Specific heat at constant volume and pressure</td>
</tr>
<tr>
<td>$C$</td>
<td>Right Cauchy–Green deformation tensor, $C = F^T \cdot F$; fourth-order elasticity tensor [see Eq. (6.3.4)]</td>
</tr>
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<td>$C_{ij}$</td>
<td>Elastic stiffness coefficients</td>
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<td>Third-order tensor of piezoelectric moduli</td>
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<td>$D$</td>
<td>Internal dissipation</td>
</tr>
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<td>$da$</td>
<td>Area element (vector) in spatial description</td>
</tr>
<tr>
<td>$dA$</td>
<td>Area element (vector) in material description</td>
</tr>
<tr>
<td>$ds$</td>
<td>Surface element in current configuration</td>
</tr>
<tr>
<td>$dS$</td>
<td>Surface element in reference configuration</td>
</tr>
<tr>
<td>$dx$</td>
<td>Line element (vector) in current configuration</td>
</tr>
<tr>
<td>$dX$</td>
<td>Line element (vector) in reference configuration</td>
</tr>
<tr>
<td>$D$</td>
<td>Symmetric part of the velocity gradient tensor, $L = (\nabla v)^T$; that is, $D = \frac{1}{2} \left[ (\nabla v)^T + \nabla v \right]$; electric flux vector; mass diffusivity tensor</td>
</tr>
<tr>
<td>$D/Dt$</td>
<td>Material time derivative</td>
</tr>
<tr>
<td>$D_i$</td>
<td>Internal diameter</td>
</tr>
<tr>
<td>$e$</td>
<td>Specific internal energy</td>
</tr>
<tr>
<td>$e$</td>
<td>Almansi strain tensor, $e = \frac{1}{2} \left( I - F^{-T} \cdot F^{-1} \right)$</td>
</tr>
<tr>
<td>$\hat{e}$</td>
<td>A unit vector</td>
</tr>
<tr>
<td>$e_A$</td>
<td>A unit basis vector in the direction of vector $A$</td>
</tr>
<tr>
<td>$e_i$</td>
<td>A basis vector in the $x_i$-direction</td>
</tr>
<tr>
<td>$e_{ijk}$</td>
<td>Components of alternating tensor, $\varepsilon$</td>
</tr>
<tr>
<td>$E$</td>
<td>Green–Lagrange strain tensor, $E = \frac{1}{2} \left( F^T \cdot F - I \right)$</td>
</tr>
<tr>
<td>$E, E_1, E_2$</td>
<td>Young’s modulus (modulus of elasticity)</td>
</tr>
<tr>
<td>$\hat{E}_i$</td>
<td>Unit base vector along the $X_i$ material coordinate direction electric field intensity vector</td>
</tr>
<tr>
<td>$E_{ij}$</td>
<td>Components of the Green–Lagrange strain tensor</td>
</tr>
<tr>
<td>$f$</td>
<td>Load per unit length of a bar</td>
</tr>
<tr>
<td>$f$</td>
<td>Body force vector</td>
</tr>
<tr>
<td>$f(\cdot)$</td>
<td>Function</td>
</tr>
<tr>
<td>$f_x, f_y, f_z$</td>
<td>Body force components in the $x$, $y$, and $z$ directions</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>F</td>
<td>Deformation gradient, $\mathbf{F} = (\nabla_0 \mathbf{x})^T$; force vector</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>Functional mapping</td>
</tr>
<tr>
<td>$g$</td>
<td>Acceleration due to gravity; function; internal heat generation</td>
</tr>
<tr>
<td>$\mathbf{g}$</td>
<td>Gradient of temperature, $\mathbf{g} = \nabla \theta$</td>
</tr>
<tr>
<td>$G$</td>
<td>Shear modulus (modulus of rigidity)</td>
</tr>
<tr>
<td>$h$</td>
<td>Height of the beam; thickness; heat transfer coefficient</td>
</tr>
<tr>
<td>$H$</td>
<td>Total entropy (see Section 5.4.3.1); unit step function</td>
</tr>
<tr>
<td>$\mathbf{H}$</td>
<td>Nonlinear deformation tensor [see Eq. 6.6.25]; magnetic field intensity vector</td>
</tr>
<tr>
<td>I</td>
<td>Second moment of area of a beam cross section; functional</td>
</tr>
<tr>
<td>$I_1, I_2, I_3$</td>
<td>Principal invariants of stress tensor</td>
</tr>
<tr>
<td>$J$</td>
<td>Determinant of the matrix of deformation gradient (Jacobian); polar second moment of area of a shaft cross section</td>
</tr>
<tr>
<td>$\mathbf{J}$</td>
<td>Current density vector; creep compliance</td>
</tr>
<tr>
<td>$J_i$</td>
<td>Principal invariants of strain tensor $\mathbf{E}$ or rate of deformation tensor $\mathbf{D}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Spring constant; thermal conductivity</td>
</tr>
<tr>
<td>$K$</td>
<td>Thermal conductivity tensor</td>
</tr>
<tr>
<td>$K_{ij}$</td>
<td>Stiffness coefficients</td>
</tr>
<tr>
<td>$K_s$</td>
<td>Shear correction factor in Timoshenko beam theory</td>
</tr>
<tr>
<td>$\ell_{ij}$</td>
<td>Direction cosines [see Eq. (2.2.71) or Eq. (4.3.4)]</td>
</tr>
<tr>
<td>$L$</td>
<td>Length; Lagrangian function</td>
</tr>
<tr>
<td>$L(\cdot)$</td>
<td>Velocity gradient tensor, $\mathbf{L} = (\nabla \mathbf{v})^T$</td>
</tr>
<tr>
<td>$[L]$</td>
<td>Linear form</td>
</tr>
<tr>
<td>$m$</td>
<td>A scalar memory function (or relaxation kernel)</td>
</tr>
<tr>
<td>$\mathbf{m}$</td>
<td>Couple traction vector [see Eq. (5.3.33)]</td>
</tr>
<tr>
<td>$M$</td>
<td>Bending moment in beam problems</td>
</tr>
<tr>
<td>$\mathbf{M}$</td>
<td>Couple stress tensor; magnetization vector</td>
</tr>
<tr>
<td>$\mathbf{n}$</td>
<td>Unit normal vector in the current configuration</td>
</tr>
<tr>
<td>$n_i$</td>
<td>$i$th component of the unit normal vector $\mathbf{n}$</td>
</tr>
<tr>
<td>$N$</td>
<td>Axial force in beam problems</td>
</tr>
<tr>
<td>$\mathbf{N}$</td>
<td>Unit normal vector in the reference configuration</td>
</tr>
<tr>
<td>$N_i$</td>
<td>$i$th component of the unit normal vector $\mathbf{N}$</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure (hydrostatic or thermodynamic)</td>
</tr>
<tr>
<td>$\mathbf{p}$</td>
<td>Angular momentum vector; vector of pyroelectric coefficients</td>
</tr>
<tr>
<td>$P$</td>
<td>Point load in beams; perimeter</td>
</tr>
<tr>
<td>$\mathbf{P}$</td>
<td>First Piola–Kirchhoff stress tensor; polarization vector</td>
</tr>
<tr>
<td>$q$</td>
<td>Distributed transverse load on a beam</td>
</tr>
<tr>
<td>$q_0$</td>
<td>Intensity of the distributed transverse load in beams</td>
</tr>
<tr>
<td>$\mathbf{q}_0$</td>
<td>Heat flux vector in the reference configuration</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

$q_n$ Heat flux normal to the boundary, \( q_n = \nabla \cdot \hat{n} \)
$q_f$ Moisture flux vector
$q_i$ Force components
$q$ Heat flux vector in the current configuration
$Q$ First moment of area; volume rate of flow
$Q$ Rotation tensor [see Eq. (3.8.12)]
$Q_h$ Heat input
$Q_J$ Joule heating
$r$ Radial coordinate in the cylindrical polar system
$r_0$ Internal heat generation per unit mass in the reference configuration
$r_h$ Internal heat generation per unit mass in the current configuration
$R$ Radial coordinate in the spherical coordinate system; universal gas constant
$R$ Position vector in the spherical coordinate system; proper orthogonal tensor
$S$ A second-order tensor; second Piola–Kirchhoff stress tensor
$S_e$ Electric susceptibility tensor
$S_{ij}$ Elastic compliance coefficients
$t$ Time
$t$ Stress vector; traction vector
$T$ Torque; temperature
$u$ Displacement vector
$v$ Velocity, \( v = |v| \)
$V$ Shear force in beam problems; potential energy due to loads
$V$ Left Cauchy stretch tensor
$W$ Power input
$W$ Skew symmetric part of the velocity gradient tensor, \( L = (\nabla v)^T \); that is, \( W = \frac{1}{2} \left[ (\nabla v)^T - \nabla v \right] \)
$x$ Spatial coordinates
$y, z$ Rectangular Cartesian coordinates
$x_1, x_2, x_3$ Rectangular Cartesian coordinates
$X$ Material coordinates
$Y$ Relaxation modulus
$z$ Transverse coordinate in the beam problem; axial coordinate in the torsion problem
LIST OF SYMBOLS

Greek symbols

\( \alpha \) Angle; coefficient of thermal expansion
\( \alpha_{ij} \) Thermal coefficients of expansion
\( \beta_{ij} \) Material coefficients, \( \beta_{ij} = C_{ijkl} \alpha_{kl} \)
\( \chi \) Deformation mapping
\( \delta \) Variational operator used in Chapter 7; Dirac delta
\( \delta_{ij} \) Components of the unit tensor, \( I \) (Kronecker delta)
\( \Delta \) Change of (followed by another symbol)
\( \varepsilon \) Infinitesimal strain tensor
\( \tilde{\varepsilon} \) Symmetric part of the displacement gradient tensor, \( (\nabla \mathbf{u})^T \); that is, \( \tilde{\varepsilon} = \frac{1}{2} \left[ (\nabla \mathbf{u})^T + \nabla \mathbf{u} \right] \)
\( \epsilon_0 \) Permittivity of free space
\( \epsilon_{ij} \) Rectangular components of the infinitesimal strain tensor
\( \phi \) A typical variable; angular coordinate in the spherical coordinate system; electric potential; relaxation function
\( \phi_f \) Moisture source
\( \Phi \) Viscous dissipation, \( \Phi = \tau : \mathbf{D} \); Gibb’s potential; Airy stress function
\( \gamma \) Shear strain in one-dimensional problems
\( \Gamma \) Internal entropy production; total boundary
\( \eta \) Entropy density per unit mass; dashpot constant
\( \eta_0 \) Viscosity coefficient
\( \kappa_0, \kappa \) Reference and current configurations
\( \lambda \) Extension ratio; Lamé constant; eigenvalue
\( \mu \) Lamé constant; viscosity; principal value of strain
\( \mu_0 \) Permeability of free space
\( \nu \) Poisson’s ratio; \( \nu_{ij} \) Poisson’s ratios
\( \Pi \) Total potential energy functional
\( \theta \) Angular coordinate in the cylindrical and spherical coordinate systems; angle; twist per unit length; absolute temperature
\( \Theta \) Twist
\( \rho \) Density in the current configuration; charge density
\( \rho_0 \) Density in the reference configuration
\( \sigma \) Boltzmann constant
\( \sigma \) Mean stress
\( \sigma \) Cauchy stress tensor
\( \tau \) Shear stress; retardation or relaxation time
\( \tau \) Viscous stress tensor
\( \Omega \) Domain of a problem
\( \Omega \) Skew symmetric part of the displacement gradient tensor, \( (\nabla \mathbf{u})^T \); that is, \( \Omega = \frac{1}{2} \left[ (\nabla \mathbf{u})^T - \nabla \mathbf{u} \right] \)
LIST OF SYMBOLS

\( \omega \) Angular velocity
\( \omega \) Infinitesimal rotation vector, \( \omega = \frac{1}{2} \nabla \times \mathbf{u} \)
\( \psi \) Warping function; stream function; creep function
\( \Psi \) Helmholtz free energy density; Prandtl stress function
\( \nabla \) Gradient operator with respect to \( x \)
\( \nabla_0 \) Gradient operator with respect to \( X \)
[ ] Matrix associated with the enclosed quantity
{ } Column vector associated with the enclosed quantity
| | Magnitude or determinant of the enclosed quantity
( ) Time derivative of the enclosed quantity
( )* Enclosed quantity with superposed rigid-body motion
( ) Deviatoric tensors associated with the enclosed tensor

Note:
Quotes by various people included in this book were found at different web sites; for example, visit:

http://naturalscience.com/dsqhome.html,
http://thinkexist.com/quotes/david_hilbert/,
and
The author cannot vouch for their accuracy; this author is motivated to include the quotes at various places in his book for their wit and wisdom.
Preface to the Second Edition

Tis the good reader that makes the good book; in every book he finds passages which seem confidences or asides hidden from all else and unmistakably meant for his ear; the profit of books is according to the sensibility of the reader; the profoundest thought or passion sleeps as in a mine, until it is discovered by an equal mind and heart.

— Ralph Waldo Emerson (1803–1882)

You cannot teach a man anything, you can only help him find it within himself.

— Galileo Galilei (1564–1642)

Engineers are problem solvers. They construct mathematical models, develop analytical and numerical approaches and methodologies, and design and manufacture various types of devices, systems, or processes. Mathematical development and engineering analysis are aids to designing systems for specific functionalities, and they involve (1) mathematical model development, (2) data acquisition by measurements, (3) numerical simulation, and (4) evaluation of the results in light of known information. Mathematical models are developed using laws of physics and assumptions concerning the behavior of the system under consideration. The most difficult step in arriving at a design that is both functional and cost-effective is the construction of a suitable mathematical model of the system’s behavior. It is in this context that a course on continuum mechanics or elasticity provides engineers with the background to formulate a suitable mathematical model and evaluate it in the context of the functionality and design constraints placed on the system.

Most classical books on continuum mechanics are very rigorous in mathematical treatments of the subject but short on detailed explanations and including few examples and problem sets. Such books serve as reference books but not as textbooks. This textbook provides illustrative examples and problem sets that enable readers to test their understanding of the subject matter and utilize the tools developed in the formulation of engineering problems.

This second edition of Introduction to Continuum Mechanics has the same objective as the first, namely, to facilitate an easy and thorough understanding of continuum mechanics and elasticity concepts. The course also helps engineers who depend on canned programs to analyze problems to interpret the results produced by such programs. The book offers a concise yet rigorous treatment of the subject of continuum mechanics and elasticity at the introductory level. In all of the chapters of the second edition, additional explanations, examples, and problems have been added. No attempt has been made to enlarge the scope or increase the number of topics covered.

The book may be used as a textbook for a first course on continuum mechanics as well as elasticity (omitting Chapter 8 on fluid mechanics and heat transfer). A solutions manual has also been prepared for the book. The solution manual is available from the publisher only to instructors who adopt the book as a textbook for a course.
Since the publication of the first edition, several users of the book communicated their comments and compliments as well as errors they found, for which the author thanks them. All of the errors known to the author have been corrected in the current edition. The author is grateful, in particular, to Drs. Karan Surana (University of Kansas), Arun Srinivasa (Texas A&M University), Rebecca Brannon (University of Utah), Vinu Unnikrishnan (University of Alabama), Wenbin Yu (Utah State University), Srikanth Vedantam (Indian Institute of Technology, Madras), Shailendra Joshi (National University of Singapore), Ganesh Subbarayan (Purdue University), S. H. Khan (Indian Institute of Technology, Kanpur), and Jaehyung Ju (University of North Texas) for their constructive comments and help. The author also expresses his sincere thanks to Mr. Peter Gordon, Senior Editor (Engineering) at Cambridge University Press, for his continued encouragement, friendship, and support in producing this book. The author requests readers to send their comments and corrections to jn_reddy@yahoo.com.

J. N. Reddy
College Station, Texas

What is there that confers the noblest delight? What is that which swells a man’s breast with pride above that which any other experience can bring to him? Discovery! To know that you are walking where none others have walked ...

— Mark Twain (1835–1910)

You can get into a habit of thought in which you enjoy making fun of all those other people who don’t see things as clearly as you do. We have to guard carefully against it.

— Carl Sagan (1934–1996)
Preface to the First Edition

If I have been able to see further, it was only because I stood on the shoulders of giants.

—– Isaac Newton (1643–1727)

Many of the mathematical models of natural phenomena are based on fundamental scientific laws of physics or otherwise, extracted from centuries of research on the behavior of physical systems under the action of natural “forces.” Today this subject is referred to simply as mechanics – a phrase that encompasses broad fields of science concerned with the behavior of fluids, solids, and complex materials. Mechanics is vitally important to virtually every area of technology and remains an intellectually rich subject taught in all major universities. It is also the focus of research in departments of aerospace, chemical, civil, and mechanical engineering, and engineering science and mechanics, as well as applied mathematics and physics. The last several decades have witnessed a great deal of research in continuum mechanics and its application to a variety of problems. As most modern technologies are no longer discipline-specific but involve multidisciplinary approaches, scientists and engineers should be trained to think and work in such environments. Therefore, it is necessary to introduce the subject of mechanics to senior undergraduate and beginning graduate students so that they have a strong background in the basic principles common to all major engineering fields. A first course on continuum mechanics or elasticity is the one that provides the basic principles of mechanics and prepares engineers and scientists for advanced courses in traditional as well as emerging fields such as biomechanics and nanomechanics.

There are many books on mechanics of continua. These books fall into two major categories: those that present the subject as a highly mathematical and abstract subject, and those that are too elementary to be of use for those who will pursue further work in fluid dynamics, elasticity, plates and shells, viscoelasticity, plasticity, and interdisciplinary areas such as geomechanics, biomechanics, mechanobiology, and nanoscience. As is the case with all other books written (solely) by the author, the objective is to facilitate an easy understanding of the topics covered. It is hoped that the book is simple in presenting the main concepts yet mathematically rigorous enough in providing the invariant form as well as component form of the governing equations for analysis of practical problems of engineering. In particular, the book contains formulations and applications to specific problems from heat transfer, fluid mechanics, and solid mechanics.

The motivation and encouragement that led to the writing of this book came from the experience of teaching a course on continuum mechanics at Virginia Polytechnic Institute and State University and Texas A&M University. A course on continuum mechanics takes different forms – from abstract to very applied – when taught by different people. The primary objective of the course taught by the author is two-fold: (1) formulation of equations that describe the motion and thermomechanical response of materials and (2) solution of these equations for specific problems from elasticity, fluid flows, and heat transfer. The present
book is a formal presentation of the author’s notes developed for such a course over the last two and half decades.

With a brief discussion of the concept of a continuum in Chapter 1, a review of vectors and tensors is presented in Chapter 2. Since the language of mechanics is mathematics, it is necessary for all readers to familiarize themselves with the notation and operations of vectors and tensors. The subject of kinematics is discussed in Chapter 3. Various measures of strain are introduced here. The deformation gradient, Cauchy–Green deformation, Green–Lagrange strain, Cauchy and Euler strain, rate of deformation, and vorticity tensors are introduced, and the polar decomposition theorem is discussed in this chapter. In Chapter 4, various measures of stress – Cauchy stress and Piola–Kirchhoff stress measures – are introduced, and stress equilibrium equations are presented.

Chapter 5 is dedicated to the derivation of the field equations of continuum mechanics, which forms the heart of the book. The field equations are derived using the principles of conservation of mass and balance of momenta and energy. Constitutive relations that connect the kinematic variables (e.g., density, temperature, and deformation) to the kinetic variables (e.g., internal energy, heat flux, and stresses) are discussed in Chapter 6 for elastic materials, viscous and viscoelastic fluids, and heat transfer.

Chapters 7 and 8 are devoted to the application of the field equations derived in Chapter 5 and constitutive models of Chapter 6 to problems of linearized elasticity, and fluid mechanics and heat transfer, respectively. Simple boundary-value problems, mostly linear, are formulated and their solutions are discussed. The material presented in these chapters illustrates how physical problems are analytically formulated with the aid of continuum equations. Chapter 9 deals with linear viscoelastic constitutive models and their application to simple problems of solid mechanics. Since a continuum mechanics course is mostly offered by solid mechanics programs, the coverage in this book is slightly more directed, in terms of the amount and type of material covered, to solid and structural mechanics.

The book was written keeping undergraduate seniors and first-year graduate students of engineering in mind. Therefore, it is most suitable as a text book for adoption for a first course on continuum mechanics or elasticity. The book also serves as an excellent precursor to courses on viscoelasticity, plasticity, nonlinear elasticity, and nonlinear continuum mechanics.

The book contains so many mathematical equations that it is hardly possible not to have typographical and other kinds of errors. I wish to thank in advance those readers who are willing to draw the author’s attention to typos and errors, using the e-mail address: jn_reddy@yahoo.com.

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About the Author

J. N. Reddy is a University Distinguished Professor, Regents Professor, and the holder of the Oscar S. Wyatt Endowed Chair in the Department of Mechanical Engineering at Texas A&M University, College Station. Prior to this current position, he was the Clifton C. Garvin Professor in the Department of Engineering Science and Mechanics at Virginia Polytechnic Institute and State University (Virginia Tech), Blacksburg.

Dr. Reddy is internationally known for his contributions to theoretical and applied mechanics and computational mechanics. He is the author of more than 480 journal papers and 18 books. Professor Reddy is the recipient of numerous awards including the Walter L. Huber Civil Engineering Research Prize of the American Society of Civil Engineers (ASCE), the Worcester Reed Richards Memorial Award of the American Society of Mechanical Engineers (ASME), the 1997 Archie Higdon Distinguished Educator Award from the American Society of Engineering Education (ASEE), the 1998 Nathan M. Newmark Medal from ASCE, the 2000 Excellence in the Field of Composites from the American Society of Composites (ASC), the 2003 Bush Excellence Award for Faculty in International Research from Texas A&M University, and the 2003 Computational Solid Mechanics Award from the U.S. Association of Computational Mechanics (USACM). Dr. Reddy received honorary degrees (Honoris Causa) from the Technical University of Lisbon, Portugal, in 2009 and Odlar Yurdu University, Baku, Azerbaijan in 2011.


Dr. Reddy is one of the selective researchers in engineering around the world who is recognized by ISI Highly Cited Researchers with more than 13,000 citations (without self-citations more than 12,000) with an h-index of more than 54 as per Web of Science, 2013; as per Google Scholar the number of citations is more than 29,000 and the h-index is 71. A more complete resume with links to journal papers can be found at

http://isihighlycited.com/ or http://www.tamu.edu/acml.