1 Gravitational attraction

1.1 Universal gravitational attraction

In his *Principia*, which he completed in 1686, Sir Isaac Newton demonstrated the inverse square law for universal gravitation. This law has provided the mathematical basis for the study of gravitational attraction among large attracting masses. Newton demonstrated that the inverse square proportionality for the attractive forces among all matter could be shown from his second law of motion and Kepler's third law of planetary motion. This gravitational force is one of the weakest forces in nature by many orders of magnitude. The sizes of the masses involved give it importance on Earth and in space. Partly because of its weakness, physicists today still argue about the role of gravity in relativity and particle physics, occasionally suggesting that at some scales the inverse square law may break down. However, for the study of the gravitational attraction of the Earth and its geological features, Newton's inverse square law and the larger field of potential theory derived from it are sufficient.

Newton's second law relates the force acting on a body to its change in momentum. In order to demonstrate the inverse square law, one can consider a point mass, m, or planet, in orbit about a larger mass, m'. The point mass experiences an attractive force pulling it back toward the larger mass (Figure 1.1). On traveling a short arc distance, c, the planet is pulled a distance, s, from its straight line path to a position closer to m'. From Newton's second law the distance, s, which is the distance a body would move under a constant acceleration, is given by the relation

$$s = \frac{1}{2}a(\delta t)^2, \tag{1.1}$$

where a is the acceleration experienced by the planet and δt is the time increment the planet takes to travel the distance c. The time increment can be eliminated by equating its fraction of the total period of revolution, T, to the ratio of the small arc distance, c, to the circumference of the orbit

$$\frac{\delta t}{T} = \frac{c}{2\pi R}. \tag{1.2}$$

Equation (1.2) allows the acceleration to be written in the form

$$\frac{a}{2} = s\left(\frac{2\pi R}{Tc}\right)^2. \tag{1.3}$$

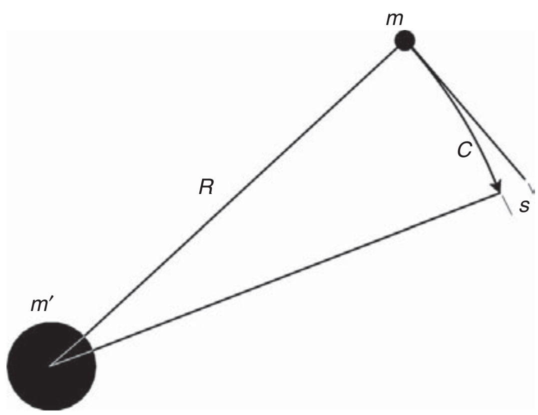

Figure 1.1 Path of particle *m* in orbit about a large planet of mass *m*′.

By a geometrical proof, Newton showed that $c^2 = 2Rs$ for arc distances c much less than the radius, R, and was able to eliminate both s and c^2 in Eq. (1.3). The resulting relation contains the square of the period of revolution. Kepler's third law, which is the observation that the squares of the periods of any two planets are proportional to the cubes of their mean distances from the Sun, provided the relation needed to demonstrate the inverse square law of universal gravitational attraction. The proportionality takes the form

$$a = 2s\left(\frac{2\pi R}{Tc}\right)^2 \geq \frac{4\pi^2 R}{T^2} \approx \frac{1}{R^2}. \tag{1.4}$$

The constant of proportionality turns out to be proportional to the attracting mass, m', and, therefore, can be written Gm', where G is the universal gravitational constant. The force of attraction between the two masses is therefore

$$F = -G\frac{m'm}{R^2}, \tag{1.5}$$

where:

$F = ma$, the magnitude of the force of attraction
$G =$ the universal gravitational constant
$m' =$ the attracting mass of the Sun
$m =$ the attracted mass of the planet
$R =$ the distance from the center of m to the center of m'

The negative sign designates the force as attractive, toward the center of m'.

The gravitational forces on m and m' are equal in magnitude, opposite in direction, and along the line joining the two masses. The force is proportional to the product of the two masses and inversely proportional to the square of their separation. The value of G based on early pendulum measurements was $(6.67428 \pm 0.00067) \times 10^{-11}$ m^3 kg^{-1} s^{-2}. The best measurements of G as of 2010 (Mohr, et al., 2011) is $(6.67384 \pm 0.00080) \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

For more than 300 years, Newton's law has provided the basis for studies of gravitational attraction, a remarkable record for a relation based on empirical observations. Recent

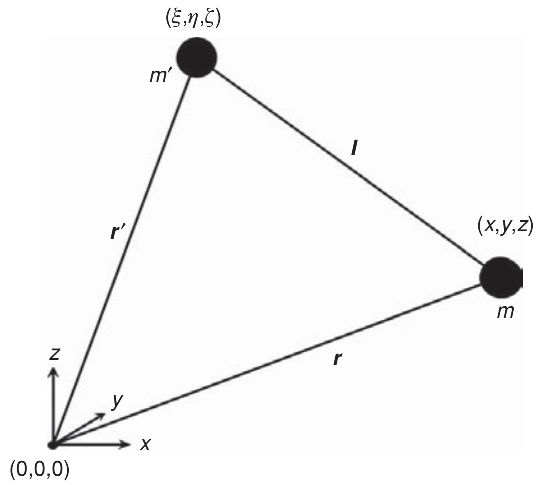

Figure 1.2 Vector notations for attraction of point masses.

speculations about a fifth force suggest a deviation from the inverse square relation, but these have proven too small, if they even exist, to be confirmed by measurements. Also, Einstein's special relativity added a new dimension to Newtonian space-time that has changed how physicists perceive gravitation. However, the relativistic deviations from Newtonian gravitation are too small to have an impact on the measurement of the Earth's gravitational attraction because the velocities in the Earth's system are insignificant relative to the speed of light. These relativistic effects have only recently become measurable in satellite data.

1.2 Gravitational acceleration

The acceleration of a small point mass near a much larger attracting mass, such as would approximate the attraction of satellites orbiting the Earth in space, can be expressed as the acceleration of a unit mass, $m = 1.0$. The magnitude, a, of the acceleration of a unit mass is

$$a = \frac{F}{m} = -G\frac{m'}{r^2}. \tag{1.6}$$

Equation (1.6) gives the gravitational attraction of the larger mass m'. Because the force of attraction and acceleration are vectors, a general expression with the origin displaced from the position of the attracting mass is more appropriate and is needed for more complex computations. The direction vector, $\boldsymbol{l}$, along which the force acts, is the difference in the position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ of the two masses (Figure 1.2). The position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ point from the origin to m and m', respectively. The gravitational attraction in vector notation may be expressed as

$$\boldsymbol{a} = -G\frac{m'(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} = -G\frac{m'}{|l|^2}\frac{\boldsymbol{l}}{|\boldsymbol{l}|}. \tag{1.7}$$

In Cartesian coordinates, the attracted mass, usually the unit test mass, m, is at the position (x, y, z), and the attracting mass is at position (ξ, η, ζ). In applications to the Earth, the origin is the center of mass of the Earth and the z-axis is the mean axis of rotation. The x- and y-axes are arbitrary, but x is by convention the meridian plane of Greenwich, England. This orientation of the reference axes defines the geocentric coordinate system for the Earth. Expanding Eq. (1.7) in terms of the coordinates of m and m' gives

$$[a_x, a_y, a_z] = -G \frac{m' [x - \xi, y - \eta, z - \zeta]}{\left((x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2\right)^{3/2}}, \tag{1.8}$$

where in Eq. (1.8), $|\boldsymbol{l}|$ has been replaced by $\sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$. The components of the vector, $\boldsymbol{a}$ in the directions of the geocentric coordinated axes are $[a_x, a_y, a_z]$.

Equation (1.8) can be modified and simplified for computation of the effects of anomalous mass near the Earth's surface. The attraction of the anomalous mass is typically less than 0.001 percent of the attraction of the Earth. Also, the curvature of the Earth's surface may be neglected for all but the larger regional surveys. Given the small magnitude of local anomalies and negligible difference between a flat plane and the survey area for local surveys, a rectangular coordinate system can be used. The vertical direction is set to coincide with the direction of the Earth's gravity field, the direction in which measurements of the magnitude of the gravity field are determined. The attraction of anomalous mass is projected onto the vertical direction and two horizontal directions (typically north and east) for computation. In most gravity measurements for the interpretation of anomalous density structures near the Earth's surface, the main portion of the Earth's field is removed and only the anomalous component is retained. In Eq. (1.8) the vertical direction corresponds to the radial or normal direction. For application to the Earth's surface in local surveys, the radial direction is usually assigned the z direction in a rectangular coordinate system. The attraction of anomalous masses at the surface, from Eq. (1.8) is

$$a_z = -G \frac{m'(z - \zeta)}{\left((x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2\right)^{3/2}}, \tag{1.9}$$

where the z-axis now refers to the vertical direction which is in line with the negative direction of the Earth's gravity field. The ratio of $z - \zeta$ to l is the cosine of the angle between vertical (the z-axis) and the attraction of anomalous mass, and thus for anomalous mass, Eq. (1.9) is the projection of the attraction of anomalous mass onto the z-, or vertical, axis.

1.3 Gravitational potential of a point mass

The gravitational vector field can be derived from a potential scalar field because the gravitational field is conservative. For a conservative vector field, the work required to move a particle from point A to B is independent of the path (Figure 1.3). The work

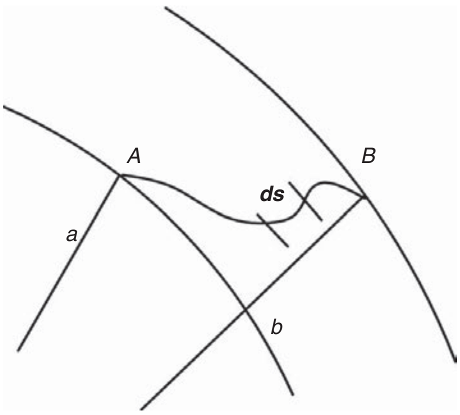

Figure 1.3 Work along path from point *A* to *B*. *A* and *B* are at different potential levels.

required, in the absence of friction, is the integral along the path of the product of force times the distance moved,

$$\Delta W = W(B) - W(A) = \int_A^B \boldsymbol{F} \cdot \boldsymbol{ds}, \tag{1.10}$$

where the dot product gives the component of the force in the direction of movement given by $\boldsymbol{ds}$. If we equate a for a point mass from Eq. (1.7) to $\boldsymbol{F}$, Eq. (1.10) becomes,

$$\Delta W = -G\int_A^B \frac{m'\boldsymbol{l} \cdot \boldsymbol{ds}}{\boldsymbol{l}^3} = -G\int_A^B \frac{m'l\cos(\theta)\,ds}{l^3} = -G\int_{l_A}^{l_B} \frac{m'dl}{l^2} = Gm'\left(\frac{1}{l_A} - \frac{1}{l_B}\right), \tag{1.11}$$

where dl is $l\cos(\theta)ds$ the projection of $\boldsymbol{ds}$ on $\boldsymbol{l}$. The gravitational potential V is the limit of ΔW as l_A goes to infinity. The expression for the potential is

$$V = -G\frac{m'}{l}. \tag{1.12}$$

In practice, the solutions for many gravity problems are easier to solve by using the scalar potential and computing the gravitational acceleration from the gradient of the potential

$$\boldsymbol{a} = -\nabla V. \tag{1.13}$$

1.4 Gravitational potential of a solid body

The gravitational attraction of a composite body, such as the Earth, is the combination of the attractions of its countless mass elements. The total attraction is the vector summation of the accelerations of the individual elements or the gradient of the scalar summation of the potentials for all the mass elements. The expression for the potential for many point masses can be written as the sum

$$V = G\frac{m_1}{l_1} + G\frac{m_2}{l_2} + G\frac{m_3}{l_3} + \cdots. \tag{1.14}$$

By letting the mass elements become smaller and more numerous, the potential for an anomalous distribution of mass can be approximated to any degree of precision. In the limit of infinitesimally small mass increments the summation can be replaced by the integral

$$V = G\int \frac{dm}{l}, \tag{1.15}$$

where dm is an infinitesimally small mass increment. The density ρ of the medium is defined as the ratio of the mass to the volume of the mass in the limit as the volume goes to zero,

$$\rho = \lim_{\delta v \to 0} \frac{m}{v}. \tag{1.16}$$

Density is a scalar function of position in an anomalous mass. The density distribution can be highly discontinuous and irregular in real materials, and may vary radically across grain boundaries, in voids, in caverns, and at the surface where rock comes in contact with the atmosphere. In practice, average or smoothed values of the density distribution are used in computation. By substituting the expression for dm in terms of density into Eq. (1.15), the integral expression for the potential is an integral over the volume,

$$V = G\int_v \frac{dm}{l} = G\int_v \frac{\rho\, dv}{l}. \tag{1.17}$$

In rectangular coordinates the potential for a solid body from Eq. (1.17) is written as

$$V(x, y, z) = G\int_v \frac{\rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{\left\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right\}^{1/2}}, \tag{1.18}$$

where $dv = d\xi\, d\eta\, d\varsigma$.

The gravitational acceleration in the three orthogonal coordinate directions, can be found by differentiating Eq. (1.18) by x, y, and z, giving, respectively,

$$a_x = \frac{\partial V}{\partial x} = -G\iiint \frac{\rho(\xi, \eta, \zeta)(x-\xi)\, d\xi\, d\eta\, d\zeta}{\left\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right\}^{3/2}} \tag{1.19}$$

$$a_y = \frac{\partial V}{\partial y} = -G\iiint \frac{\rho(\xi, \eta, \zeta)(y-\eta)\, d\xi\, d\eta\, d\zeta}{\left\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right\}^{3/2}} \tag{1.20}$$

$$a_z = \frac{\partial V}{\partial z} = -G\iiint \frac{\rho(\xi, \eta, \zeta)(z-\zeta)\, d\xi\, d\eta\, d\zeta}{\left\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right\}^{3/2}}. \tag{1.21}$$

In computing the gravitational attraction, it is often easier to integrate the potential once and differentiate than it is to solve the integration three times.

The attraction between two static rigid bodies, each too irregular in shape to assume they are equivalent to a point mass, requires integrations over the total volume of both masses. For example, the gravitational attraction in the x direction, for the center of mass of two irregular bodies is given by the six integrations

$$a_x = G \int\limits_{\xi_1}\int\limits_{\eta_1}\int\limits_{\zeta_1}\int\limits_{\xi_2}\int\limits_{\eta_2}\int\limits_{\zeta_2} \frac{\rho_1(\xi_1,\eta_1,\zeta_1)\,\rho_2(\xi_2,\eta_2,\zeta_2)\,(\xi_1-\xi_2)\,d\xi_1 d\eta_1 d\zeta_1 d\xi_2 d\eta_2 d\zeta_2}{\left\{(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2\right\}^{3/2}}. \tag{1.22}$$

For moving systems the asymmetry of mass can lead to rotational forces, such as those exerted by the Sun and Moon on the bulge of the rotating Earth. These forces create a torque that explains the precession of the Earth's axis of rotation.

1.5 Surface potential

When the distribution of mass is restricted to a thin two-dimensional sheet, it is convenient to express the equation for the potential as a two-dimensional integral over that surface. For computation, the thickness is assumed to go to zero and the sheet has a mass density of $\kappa = dm/ds$, defined by the ratio of mass, dm, to surface area, ds. In practice, the sheet only needs to be thin relative to the distance of the attracted point for the computation to be useful in modeling. In this case the surface density is the limit as the thickness goes to zero of the product of density, ρ, and thickness, dh,

$$\kappa = \frac{dm}{ds} \cong \rho dh. \tag{1.23}$$

The integral for the potential can be expressed in various forms, such as

$$V = G\iiint\limits_v \frac{dm}{l} = G\iiint\limits_v \frac{\rho dv}{l} = G\iiint\limits_v \frac{\rho dh ds}{l} = G\iiint\limits_v \frac{\kappa ds}{l}. \tag{1.24}$$

On the surface, the normal derivatives are discontinuous. They differ according to whether the derivative is taken from the internal or external side of the surface,

$$\left.\frac{\partial V}{\partial n}\right|_e = -2\pi G\kappa + G\iint\limits_\sigma \kappa \frac{\partial}{\partial n}\left(\frac{1}{l}\right) d\sigma \tag{1.25}$$

$$\left.\frac{\partial V}{\partial n}\right|_i = +2\pi G\kappa + G\iint\limits_\sigma \kappa \frac{\partial}{\partial n}\left(\frac{1}{l}\right) d\sigma \tag{1.26}$$

$$\left.\frac{\partial V}{\partial n}\right|_i - \left.\frac{\partial V}{\partial n}\right|_e = 4\pi G\kappa. \tag{1.27}$$

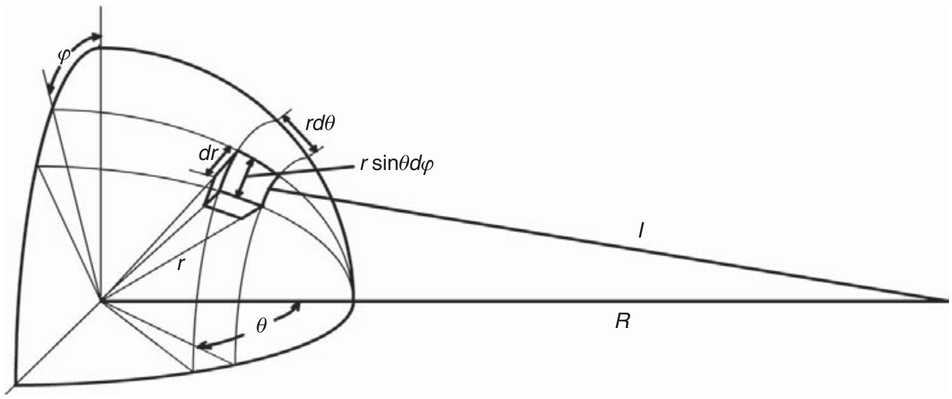

Figure 1.4 Coordinate system for computation of the potential for a constant density sphere.

1.6 Attraction of a sphere

The attraction of a sphere of uniform density is, perhaps, the most useful and fundamental relation for the interpretation of gravity anomalies. It is a first approximation to the attraction of any compact irregularly shaped body of mass at distances that are greater than the diameter of the body. In order to demonstrate the integration, Eq. (1.18) is expressed in spherical coordinates in which the incremental volume is $r^2 \sin\theta drd\theta d\phi$, where the coordinates are defined in Figure 1.4. The expression for the potential in spherical coordinates is

$$V = G\rho V = G\rho \int_0^a \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\theta}{l} drd\theta d\phi. \tag{1.28}$$

The integration over the shell is computed first. The integration over the longitudinal coordinate, λ, is trivial because the distance, l does not change with a change in ϕ, and the integration simply gives the factor of 2π,

$$V = 2\pi G\rho \int_0^a \int_0^\pi \frac{r^2 \sin\theta}{l} drd\theta. \tag{1.29}$$

The distance l is a function of a, but it can be expressed in spherical coordinates through the law of cosines for the triangle formed by the origin, the attracted mass and the incremental attracting mass

$$l^2 = r^2 + R^2 + 2rR\cos\theta. \tag{1.30}$$

By differentiating Eq. (1.30) with respect to θ it can be shown that

$$\frac{dl}{rR} = \frac{\sin\theta}{l} d\theta. \tag{1.31}$$

This relation may be substituted back into Eq. (1.29) to change the variable of integration,

$$V = 2\pi G\rho \int_0^a \int_{R-r}^{R+r} \frac{r^2}{rR} dr\,dl. \tag{1.32}$$

Evaluation of the integral for l gives

$$V = 2\pi G\rho \int_0^a \frac{r^2 dr}{rR} l\Big|_{R-r}^{R+r} = 2\pi G\rho \int_0^a \frac{r^2 dr}{rR} \left[(R+r) - (R-r)\right], \tag{1.33}$$

or

$$V = -4\pi G\rho \int_0^a \frac{r^2 dr}{R} = -\frac{4\pi}{3} a^3 G\rho \frac{1}{R}. \tag{1.34}$$

The force of attraction of the sphere in the radial direction is the derivative with respect to R, or

$$a_x = -\frac{\partial V}{dR} = \frac{4\pi}{3} a^3 G\rho \frac{2}{R^2}. \tag{1.35}$$

This radial direction is referenced to the center of mass of the sphere. The integration has demonstrated that the attraction of a sphere is equivalent to the attraction from a point at the center of the sphere with all the mass concentrated at the center.

For a small spherical zone of anomalous density near the Earth's surface, the anomalous field due to the sphere will be proportional to the density contrast between the sphere and the Earth. Because the Earth's gravity field is on the order of 6 orders of magnitude greater than anomalous fields, the anomalous attraction of a spherical shaped anomaly can be measured in the vertical direction. As in Eq. (1.21), the vertical component of the attraction of a sphere is given by the derivative of Eq. (1.34) with respect to the vertical, z,

$$a_z = \frac{\partial V}{\partial z} = \frac{4\pi}{3} a^3 G\Delta\rho \frac{(z-\zeta)}{\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{3/2}}. \tag{1.36}$$

1.7 Units of acceleration

The S.I. units of $\boldsymbol{g}$ are m/s^2, although other units are still frequently used. The practical unit for measurements of variations in the Earth's gravity is on the order of μm/s^2. One μm/s^2 corresponds to the "gravity unit" or "g.u." originally used in oil exploration geophysics. Although S.I. units are preferred or required by most journals, some recent literature in the field of geophysics still uses the Gal, one cm/s^2, for presentations of gravity data. In these publications the mGal = 0.001Gal = 10 μm/s^2 is the most common unit for contouring gravity data.

2 Instruments and data reduction

2.1 The gravitational constant

The gravitational attraction of the Earth according to Newton's universal law of gravitation is proportional to the mass of the Earth and inversely proportional to the distance from the Earth's center. The constant of proportionality is the universal gravitational constant, G. In measuring gravitational attraction the mass of any planetary body and G are coupled and planetary observations cannot be used to determine the independent values of G and mass. Independent measurements of G and the mass of the Earth, or equivalently its mean density, have not been easy. The value of G is currently defined to four significant figures,

$$G = (6.67384 \pm 0.00080) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}, \tag{2.1}$$

or 10^{-8} dyne cm^2/g^2. This most recent value of G is from the 2010 CODATA Recommended Values for Physical Constants, which were released June 2011 by the National Institute of Standards and Technology. Physicists still argue about the meaning of gravity and whether G is truly a constant; but, for the practical study and measurement of the Earth's shape and its structures, the implications of these arguments are insignificant.

The first experiment capable of determining the universal gravitational constant was carried out in 1798 by Cavendish. The Cavendish apparatus, a torsion balance, made use of the attraction of spheres, where the attraction of a sphere is known to be the same as the attraction of a point mass at the center of the sphere. A large mass, M in Figure 2.1, is moved into position and the deflection, ε, of a torsion balance is observed. The small mass and elastic constant can be expressed in terms of the period of the torsion balance without the large mass. Using the ratio of the size of the two masses and the attraction of one mass to the Earth, Cavendish was able to compute the Earth's mean density. Actual computations of G came much later from similar measurements. The more accurate recent measurements determine the period with and without the larger mass. The improvement in the recent measurements comes from improvements in the ability to measure period more accurately and from the increased precision in displacement afforded with laser interferometers. Parks and Faller (2010) used a laser interferometer to measure the change in spacing between two free-hanging pendulum masses to provide one of the most recent and precise measurements of G.