Differential Geometry of Singular Spaces and Reduction of Symmetry

In this book, the author illustrates the power of the theory of subcartesian differential spaces for investigating spaces with singularities. Part I gives a detailed and comprehensive presentation of the theory of differential spaces, including integration of distributions on subcartesian spaces and the structure of stratified spaces. Part II presents an effective approach to the reduction of symmetries.

Concrete applications covered in the text include the reduction of symmetries of Hamiltonian systems, non-holonomically constrained systems, Dirac structures and the commutation of quantization with reduction for a proper action of the symmetry group. With each application, the author provides an introduction to the field in which relevant problems occur.

This book will appeal to researchers and graduate students in mathematics and engineering.

J. Śniatycki is a Professor in the Department of Mathematics and Statistics at the University of Calgary.

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J. ŚNIATYCKI

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada

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Preface

My first encounter with differential spaces was in the mid 1980s. At a conference in Toruń, I presented the notion of algebraic reduction of symmetries of a Hamiltonian system. After the lecture, Constantin Piron asked me if my reduced spaces were the differential spaces of Sikorski. I had to admit that I did not know what Sikorski's differential spaces were. To this Piron replied something like 'You should be ashamed of yourself! You are a Pole and you do not know what are differential spaces of Sikorski!' During the lunch break I went to the library to consult Sikorski's work. In the afternoon session, I told Piron that the spaces we were dealing with were not the differential spaces of Sikorski. At that time I did not realize that they were differential schemes.

Around the same time, Richard Cushman was working out his examples of singular reduction. I was fascinated by his pictures of reduced spaces with singularities. However, I had not the faintest idea what he was really doing. Since Richard was spending a lot of time in Calgary working on his book with Larry Bates, I had a chance to ask him to explain singular reduction to me. It took me a long time to realize that he was talking the language of differential spaces without being aware of it. From conversations with Richard, it became clear that differential spaces provided a convenient language for the description of the reduction of symmetries for proper actions of symmetry groups.

The next push in the direction of serious investigations of differential spaces came from Ryer Sjamaar and Eugene Lerman. In their *Annals of Mathematics* paper on reduction of symmetries of Hamiltonian systems, they proved a theorem using techniques that are natural to the theory of differential spaces. Studying their proof, I realized that it was very simple and that I could not think of an equally simple proof that would not utilize their techniques. It convinced me that the language of differential spaces facilitated obtaining new results, and I decided to investigate if reduction of symmetries could be completely formulated and analysed within the category of differential spaces.

The theory of differential spaces is essentially differential geometry not restricted to smooth manifolds. Roman Sikorski, who is considered the father of the theory, called his book (in Polish) *Wstęp do Geometrii Różniczkowej*. This translates as 'Introduction to Differential Geometry'. Originally, differential geometry meant the description, in terms of differentiable functions, of curves and surfaces in $\mathbb{R}^n$. Singularities of curves or surfaces under consideration could also be described in terms of smooth functions. Differential geometry evolved in two different directions: the theory of manifolds and singularity theory. Manifolds are smooth spaces not presented as subsets of $\mathbb{R}^n$. Singularity theory is the study of the failure of the manifold structure. Differential geometry in the sense of Sikorski is a reunification of the two theories. It contains the theory of manifolds and also allows the investigation of singularities. It is the investigation of geometry in terms of differentiable functions. Differential geometry, understood in this way, is analogous to algebraic geometry, which is the investigation of geometry in terms of polynomials. The difference between the two theories is in the choice of the space of functions.

I am grateful to Constantin Piron for drawing my attention to Sikorski's book. I greatly appreciate the support and encouragement of Hans Duistermaat. I would like to thank Larry Bates for his support and for bringing Richard Cushman to Calgary, and to thank Jordan Watts for his interest in my work. Above all, I want to thank Richard Cushman for his patience in explaining to me the foundations of his theory of singular reduction and his subsequent collaboration, encouragement and criticism. I also want to thank Cathy Beveridge and Leslie McNab for their help in editing the manuscript. Both Cathy and Leslie have worked hard to make sure that this book is written in proper English. However, I am sure that, in spite of their vigilance, I will have managed to slip in some phrases that go against the proper use of English. Last but not least, I want to thank my wife, Pamela Plummer, without whose support this book would not have been possible.

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List of selected symbols

Upper case Latin alphabet

Ad^*	co-adjoint action
B	open ball
$\mathbb{C}$	complex numbers
$\mathcal{C}^0(S)$	continuous functions on S
$\mathcal{C}^\infty(S)$	smooth functions on S
$\mathcal{C}^\infty(S)^G$	G-invariant functions on S
D	distribution; real part of polarization in Chapter 7
$\mathcal{D}$	space of compactly supported sections
$\mathcal{D}'$	dual of $\mathcal{D}$
$\mathrm{Der}\,\mathcal{C}^\infty(S)$	space of derivations of $\mathcal{C}^\infty(S)$
$\mathfrak{E}$	family of Hamiltonian vector fields
$\mathrm{Exp} : T_pP \to P$	exponential map defined by connection
F	function; polarization in Chapter 7
$\mathcal{F}$	family of functions
$\mathfrak{F}$	family of vector fields
$\mathcal{F}F$	bundle of linear frames of F
$\mathcal{F}T^{\mathbb{C}}M$	bundle of linear frames of $T^{\mathbb{C}}M$
G	Lie group
H	Lie group
$\mathcal{H}$	Hilbert space
I	interval; inclusion map of co-adjoint orbit into $\mathfrak{g}^*$ in Chapter 7
J	momentum map
$\mathcal{J}_0$	ideal generated by components of J
K	manifold
L	manifold
$\mathcal{L}$	prequantization line bundle
M	manifold; stratum

$\mathfrak{M}$	stratification
N	manifold, stratum
N^H	normalizer of H
$\mathcal{N}(S)$	space of functions with vanishing restrictions to S
$\mathfrak{N}$	stratification
O	orbit
$\mathfrak{O}$	partition by orbits
P	manifold, differential space
$\boldsymbol{P}$	prequantization map
P^H	set of points in P fixed by action of H
P_H	set of points in P of symmetry type H
$P_{(H)}$	set of points in P of orbit type H
$\mathfrak{P}(R)$	Poisson vector fields on R
Q	manifold
$\boldsymbol{Q}$	quantization map
R	manifold; differential space; orbit space
$\mathbb{R}$	real numbers
$\boldsymbol{R}$	linear representation
$\mathcal{R}(S)$	space of restrictions to S of functions defined on a larger space
S	manifold; differential space
S_p	slice at p
$\mathcal{S}^\infty(\mathcal{L})$	space of smooth sections of $\mathcal{L}$
$\mathcal{S}^\infty_F(\mathcal{L})$	space of smooth polarized sections of $\mathcal{L}$
$\mathcal{S}^\infty(\mathcal{L})^G$	space of G-invariant smooth sections of $\mathcal{L}$
$T^{\mathbb{C}}M$	complexified tangent bundle of M
$TS,\ T^*S$	tangent and cotangent bundle spaces of S
$T\varphi$	derived map of φ
$T_p^\perp L$	symplectic complement of T_pL
U, V, W	open subsets
$\boldsymbol{U}$	unitary representation
X, Y, Z	global derivations; vector fields; sections of tangent bundle
$X(f)$	evaluation of X on f
$\mathfrak{X}(S)$	family of all vector fields on S

Lower case Latin alphabet

a, b	real numbers
c	complex number
$c : I \to S$	curve in S
d	differential

e	group identity
$\exp : \mathfrak{g} \to G$	exponential map
$\exp(tX)$	local one-parameter local group of diffeomorphisms defined by X
$\exp(tX)(x)$	point on maximal integral curve of X through x
f	function
$f^{-1}(I)$	inverse image of I under f
g	element of group G
$\mathfrak{g}$	Lie algebra of G
$\mathfrak{g}^*$	dual of $\mathfrak{g}$
h	function
$\hbar$	Planck's constant divided by 2π
$\mathfrak{h}$	Lie algebra of H
$\mathfrak{h}^*$	dual of $\mathfrak{h}$
$\mathrm{hor}\, TP$	horizontal distribution on P
$\boldsymbol{k}$	Riemannian metric
$\mathfrak{m}$	Lie algebra
$\mathfrak{n}$	Lie algebra of N
p	point
q	point
s	parameter
$\mathrm{supp}\, f$	support of f
t	parameter
u, v	derivation at a point; vector
$\mathrm{ver}\, TP$	vertical distribution on P
w	derivation at a point; vector
x, y	point
z	point of $\mathcal{L}$

Lower case Greek alphabet

α	1-form
β	1-form
δ_{ij}	Kronecker δ
ζ, η	elements of Lie algebra
θ	1-form
ϑ	cotangent bundle projection
λ, μ, ν	elements of co-adjoint orbit
$\lambda : \mathcal{L} \to P$	complex line bundle projection
ξ	element of Lie algebra

π	projection map
ϖ	form; distributional symplectic form in Chapter 8
ρ	map
ρ^*	pull-back by ρ
ρ_*	push-forward by ρ
σ	section
σ^*	pull-back by σ
σ_*	push-forward by σ
τ	map; tangent bundle projection
τ^*	pull-back by τ
φ	map
φ^*	pull-back by φ
φ_*	push-forward by φ
ω	symplectic form

Upper case Greek alphabet

Λ	Lagrangian submanifold
Π	projection of G-invariant section
Σ	restriction of section
Φ	action
Ψ	action
Ω	symplectic form of co-adjoint orbit
$\mathbf{\Omega}^k_K(S)$	space of Koszul k-forms on S
$\mathbf{\Omega}^k_M(S)$	space of Marshall k-forms on S
$\mathbf{\Omega}^k_Z(S)$	space of Zariski k-forms on S

Non-alphabetic symbols

∇	covariant derivative
$\langle \cdot \mid \cdot \rangle$	evaluation; sesquilinear form on a line bundle
$\sqrt{\lvert \wedge^n F \rvert}$	half-densities on F
$\lrcorner$	left interior product
$[\cdot, \cdot]$	Lie bracket
$\{\cdot, \cdot\}$	Poisson bracket
$\vert$	restriction
$(\cdot \mid \cdot)$	scalar product on a Hilbert space